10. When a system vibrates in a fluid medium, the damping is
(a) viscous (b) Coulomb (c) solid

11. When parts of a vibrating system slide on a dry surface, the damping is
(a) viscous (b) Coulomb (c) solid

12. When the stress-strain curve of the material of a vibrating system exhibits a hysteresis loop, the damping is
(a) viscous (b) Coulomb (c) solid

1.7 Determine the equivalent spring constant of the system shown in Fig. 1.67.

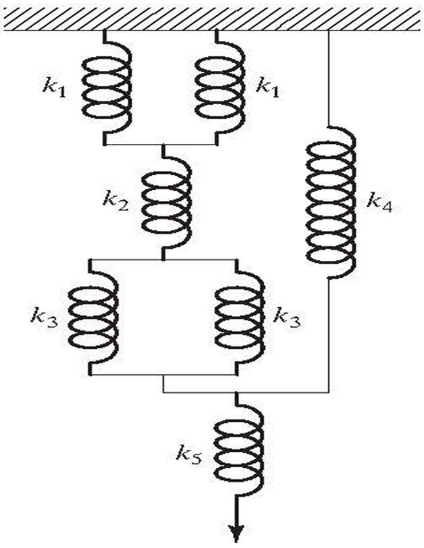
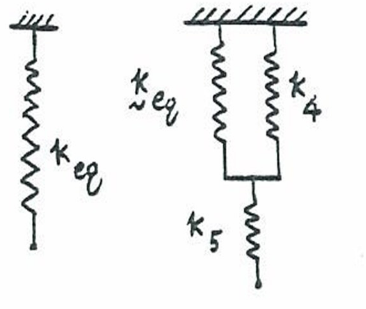


FIGURE 1.67 Springs in series-parallel.

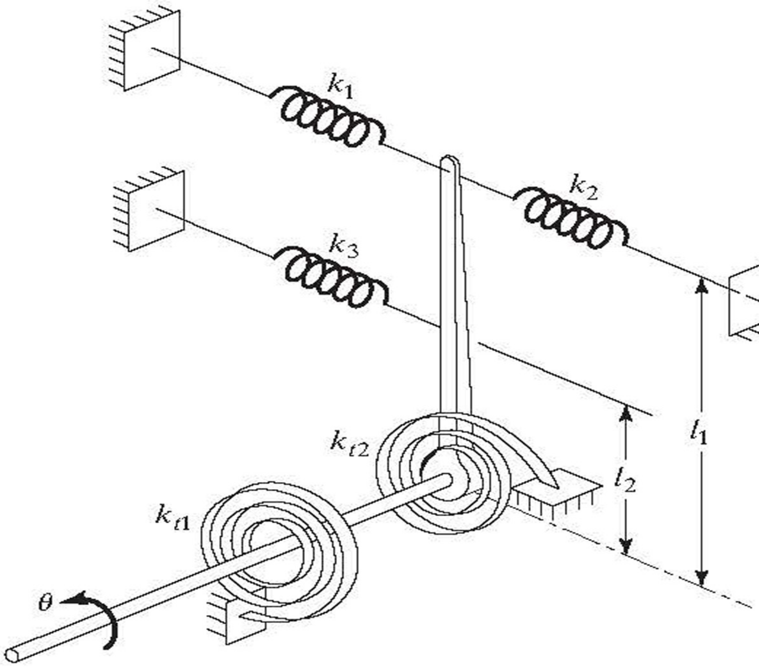
$$\frac{1}{\tilde{k}_{eq}} = \frac{1}{2k_1} + \frac{1}{k_2} + \frac{1}{2k_3} \quad ; \quad \tilde{k}_{eq} = \left(\frac{2k_1 k_2 k_3}{k_2 k_3 + 2k_1 k_3 + k_1 k_2} \right)$$



$$\frac{1}{k_{eq}} = \frac{1}{\tilde{k}_{eq} + k_4} + \frac{1}{k_5}$$

$$k_{eq} = \frac{k_5 (\tilde{k}_{eq} + k_4)}{k_5 + k_4 + \tilde{k}_{eq}} = \frac{k_2 k_3 k_4 k_5 + 2k_1 k_3 k_4 k_5 + k_1 k_2 k_4 k_5 + 2k_1 k_2 k_3 k_5}{k_2 k_3 k_4 + k_2 k_3 k_5 + 2k_1 k_3 k_4 + 2k_1 k_3 k_5 + k_1 k_2 k_4 + k_1 k_2 k_5 + 2k_1 k_2 k_3}$$

1.9 In Fig. 1.69, find the equivalent spring constant of the system in the direction of θ.



Equivalence of potential energies gives

$$\frac{1}{2} k_{t1} \theta^2 + \frac{1}{2} k_{t2} \theta^2 + \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_1)^2 + \frac{1}{2} k_3 (\theta l_2)^2 = \frac{1}{2} k_{eq} \theta^2$$

$$\therefore \; k_{eq} = k_{t1} + k_{t2} + k_1 l_1^2 + k_2 l_1^2 + k_3 l_2^2$$

1.10 Find the equivalent torsional spring constant of the system shown in Fig. 1.70. Assume that k_1, k_2, k_3, and k_4 are torsional and k_5 and k_6 are linear spring constants.

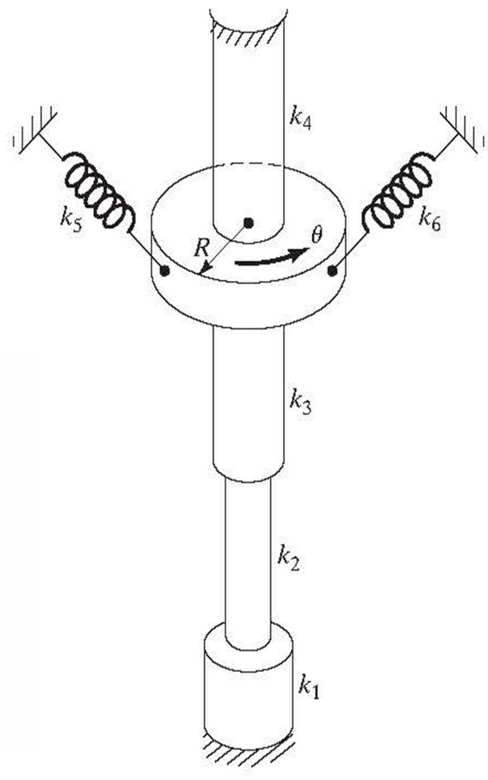


FIGURE 1.70

k_{123} = for series springs k_1, k_2 and k_3:

$$\frac{1}{k_{123}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \; ; \qquad k_{123} = \frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

Using energy equivalence,

$$\frac{1}{2} k_{eq} \theta^2 = \frac{1}{2} k_4 \theta^2 + \frac{1}{2} k_{123} \theta^2 + \frac{1}{2} k_5 (\theta R)^2 + \frac{1}{2} k_6 (\theta R)^2$$

$$\therefore \; k_{eq} = k_4 + k_{123} + R^2 k_5 + R^2 k_6$$

$$= k_4 + \left(\frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1} \right) + R^2 (k_5 + k_6)$$

1.18 The static equilibrium position of a massless rigid bar, hinged at point O and connected with springs k_1 and k_2, is shown in Fig. 1.74. Assuming that the displacement (x) resulting from the application of a force F at point A is small, find the equivalent spring constant of the system, k_e, that relates the applied force F to the displacement x as $F = k_e x$.

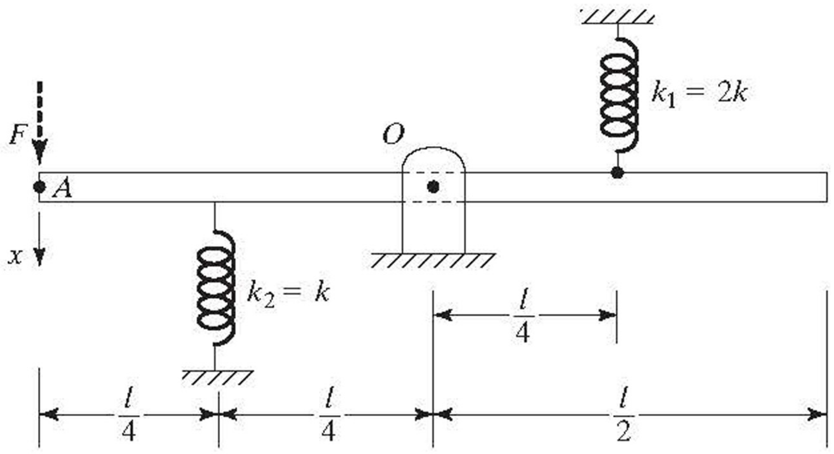
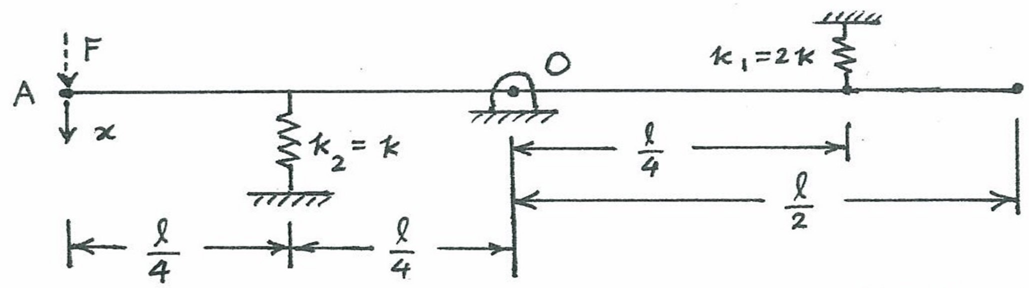
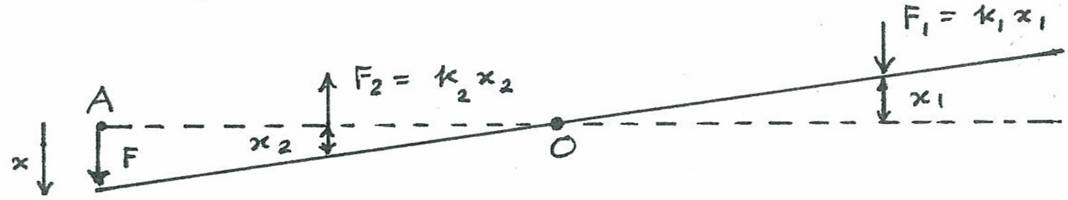


FIGURE 1.74 Rigid bar connected by springs.





$$x_2 = \frac{x}{2}, \quad x_1 = \frac{x}{2}$$

$$F_2 = k_2 x_2 = \frac{kx}{2}, \quad F_1 = k_1 x_1 = 2k\left(\frac{x}{2}\right) = kx$$

Equivalent spring constant of the system (k_{eq}) at point A can be determined by considering the moment equilibrium of forces about the pivot point O:

$$F\left(\frac{l}{2}\right) - F_2\left(\frac{l}{4}\right) - F_1\left(\frac{l}{4}\right) = 0$$

$$F = \frac{F_2}{2} + \frac{F_1}{2} = \frac{kx}{4} + \frac{kx}{2} = \frac{3}{4}kx$$

$$= k_{eq}\, x$$

$$\therefore \; k_{eq} = \frac{3}{4}k$$

1.20 Figure 1.76 shows a uniform rigid bar of mass m that is pivoted at point O and connected by springs of stiffnesses k_1 and k_2. Considering a small angular displacement θ of the rigid bar about the point O, determine the equivalent spring constant associated with the restoring moment.

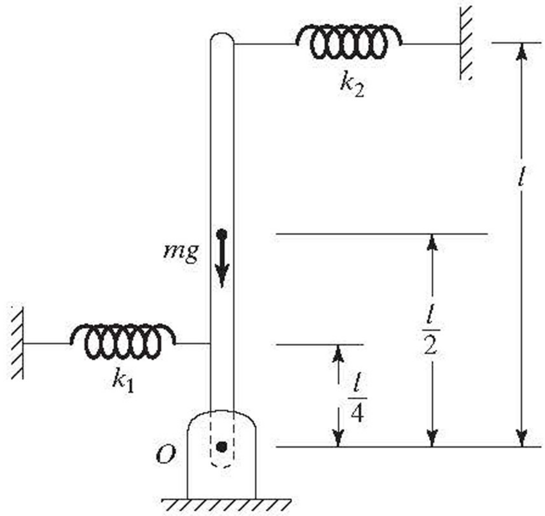


FIGURE 1.76 Rigid bar connected by springs.

1.26 Find the equivalent spring constant of the system shown in Fig. 1.82.

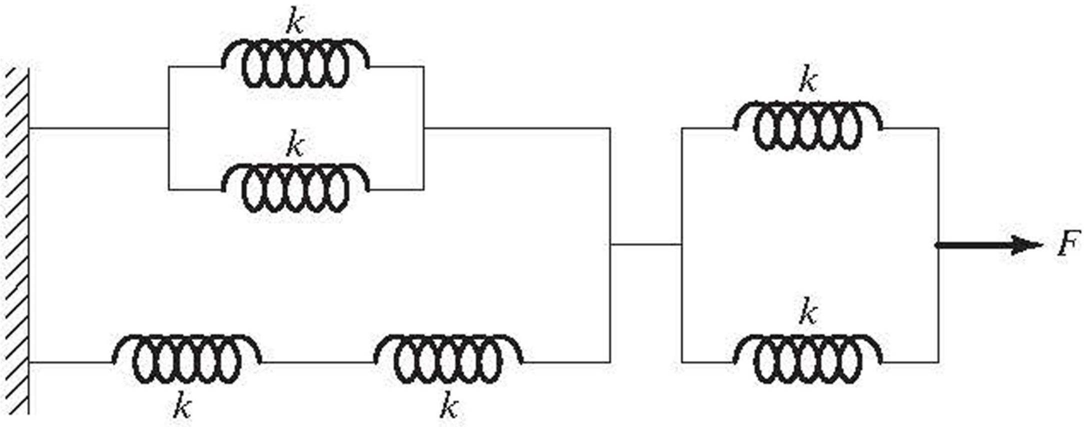


FIGURE 1.82 Springs connected in series-parallel

1.27 Figure 1.83 shows a three-stepped shaft fixed at one end and subjected to a torsional moment T at the other end. The length of step i is l_i and its diameter is D_i, $i = 1, 2, 3$. All the steps are made of the same material with shear modulus $G_i = G$, $i = 1, 2, 3$.

a. Find the torsional spring constant (or stiffness) k_{t_i} of step i ($i = 1, 2, 3$).

b. Find the equivalent torsional spring constant (or stiffness) of the stepped shaft, $k_{t_{eq}}$, so that $T = k_{t_{eq}}\theta$.

c. Indicate whether the steps behave as series or parallel torsional springs.

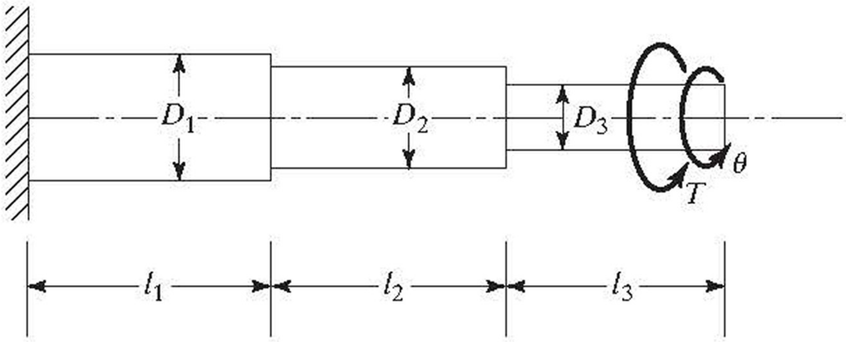


FIGURE 1.83 A stepped shaft subjected to torsional moment.

1.31 Derive the expression for the equivalent spring constant that relates the applied force F to the resulting displacement x of the system shown in Fig. 1.86. Assume the displacement of the link to be small.

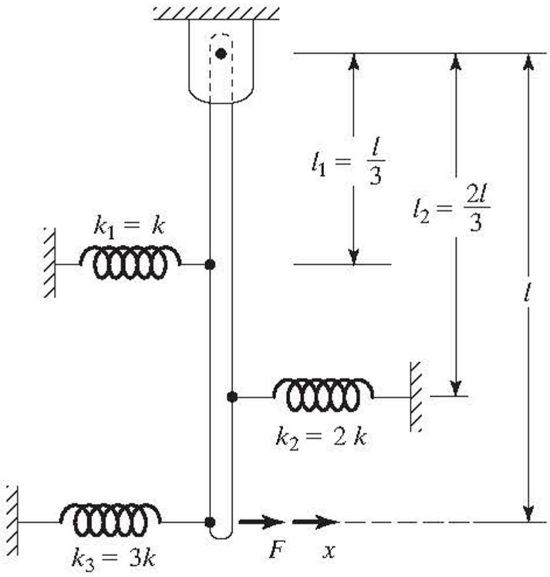


FIGURE 1.86 Rigid bar connected by springs.

1.51 Two masses, having mass moments of inertia J_1 and J_2, are placed on rotating rigid shafts that are connected by gears, as shown in Fig. 1.98. If the numbers of teeth on gears 1 and 2 are n_1 and n_2, respectively, find the equivalent mass moment of inertia corresponding to θ_1.

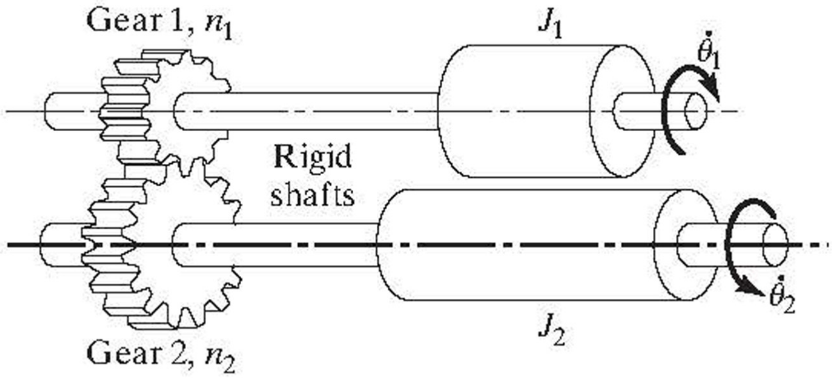


FIGURE 1.98 Rotational masses on geared shafts.

1.53 Find the equivalent mass of the system shown in Fig. 1.100.

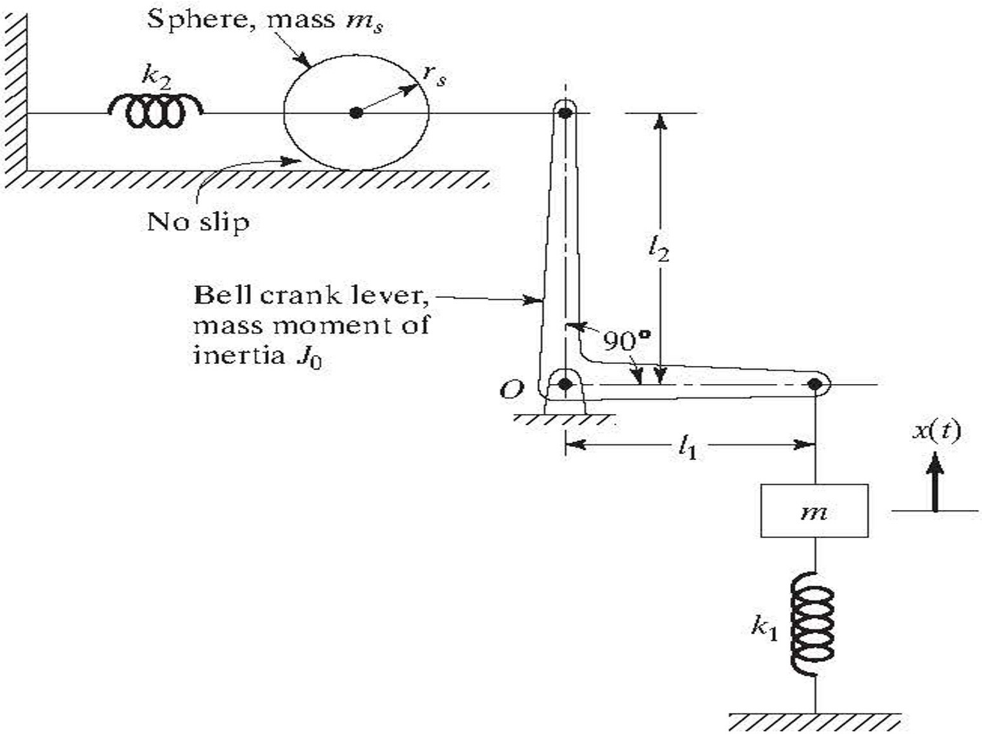


FIGURE 1.100

EXAMPLE 1.5 Equivalent k of a Suspension System

Figure 1.29 shows the suspension system of a freight truck with a parallel-spring arrangement. Find the equivalent spring constant of the suspension if each of the three helical springs is made of steel with a shear modulus $G = 80 \times 10^9$ N/m^2 and has five effective turns, mean coil diameter $D = 20$ cm, and wire diameter $d = 2$ cm.

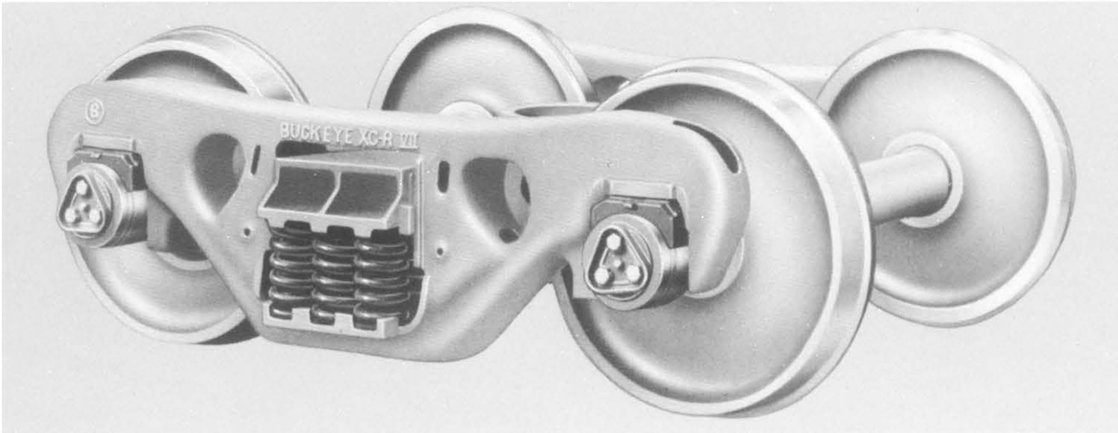


FIGURE 1.29 Parallel arrangement of springs in a freight truck. (Courtesy of Buckeye Steel Castings Company.)

EXAMPLE 1.6

Torsional Spring Constant of a Propeller Shaft

Determine the torsional spring constant of the steel propeller shaft shown in Fig. 1.30.

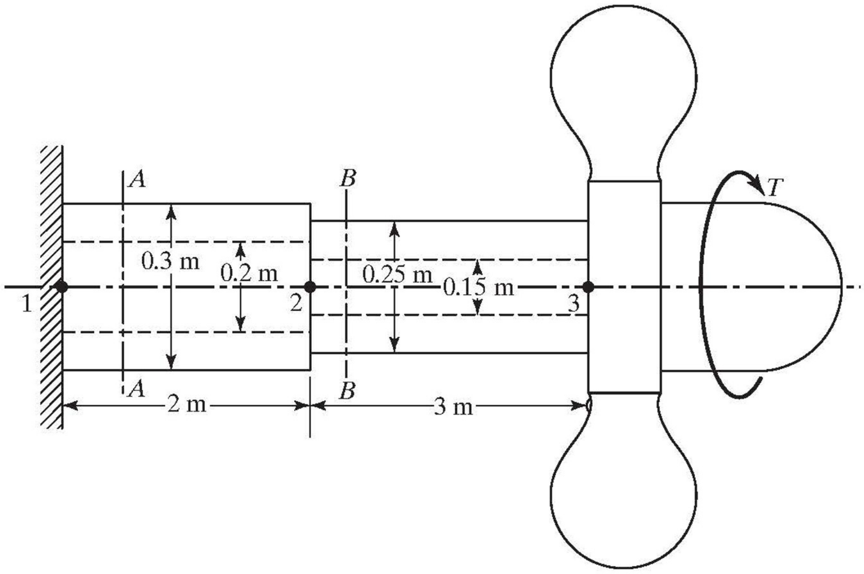


FIGURE 1.30 Propeller shaft.

EXAMPLE 1.11 Equivalent Mass of a System

Find the equivalent mass of the system shown in Fig. 1.38, where the rigid link 1 is attached to the pulley and rotates with it.

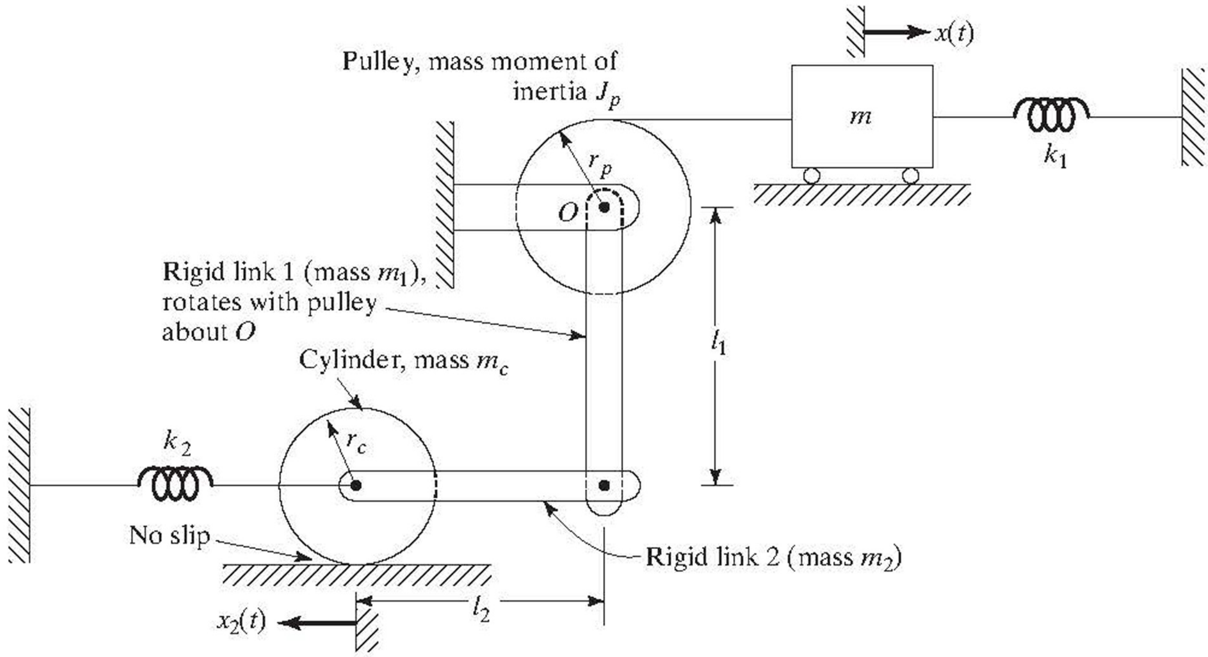


FIGURE 1.38 System considered for finding equivalent mass.

EXAMPLE 1.12 Cam-Follower Mechanism

A cam-follower mechanism (Fig. 1.39) is used to convert the rotary motion of a shaft into the oscillating or reciprocating motion of a valve. The follower system consists of a pushrod of mass m_p, a rocker arm of mass m_r, and mass moment of inertia J_r about its C.G., a valve of mass m_r, and a valve spring of negligible mass [1.28–1.30]. Find the equivalent mass (m_{eq}) of this cam-follower system by assuming the location of m_{eq} as (i) point A and (ii) point C.

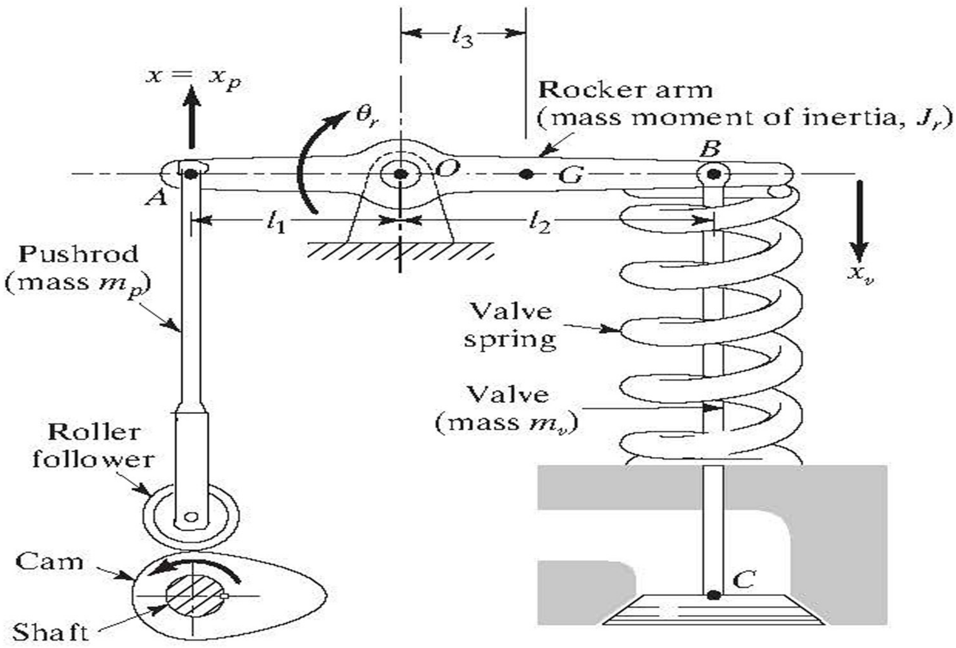


FIGURE 1.39 Cam-follower system.

EXAMPLE 2.1

Harmonic Response of a Water Tank

The column of the water tank shown in Fig. 2.10(a) is 300 ft high and is made of reinforced concrete with a tubular cross section of inner diameter 8 ft and outer diameter 10 ft. The tank weighs 6×10^5 lb when filled with water. By neglecting the mass of the column and assuming the Young's modulus of reinforced concrete as 4×10^6 psi, determine the following:

a. the natural frequency and the natural time period of transverse vibration of the water tank.
b. the vibration response of the water tank due to an initial transverse displacement of 10 in.
c. the maximum values of the velocity and acceleration experienced by the water tank.

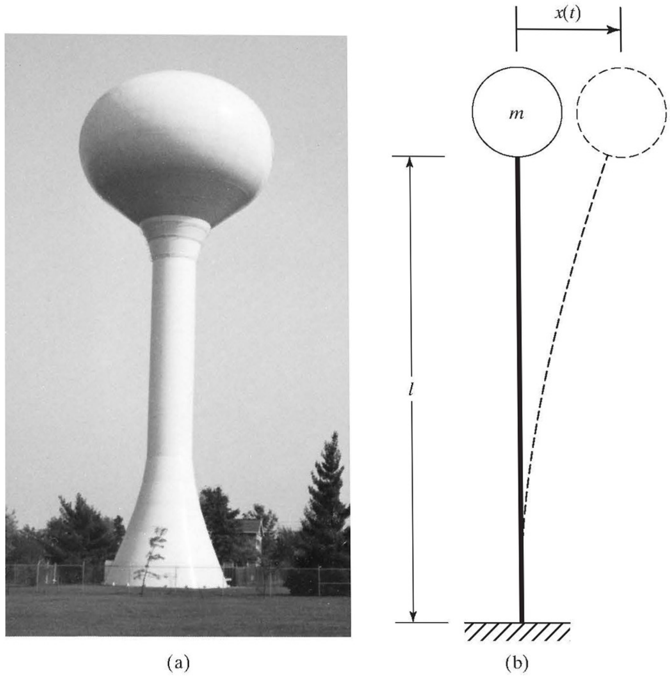


FIGURE 2.10 Elevated tank. (Photo courtesy of West Lafayette Water Company.)

EXAMPLE 2.2

Free-Vibration Response Due to Impact

A cantilever beam carries a mass M at the free end as shown in Fig. 2.11(a). A mass m falls from a height h onto the mass M and adheres to it without rebounding. Determine the resulting transverse vibration of the beam.

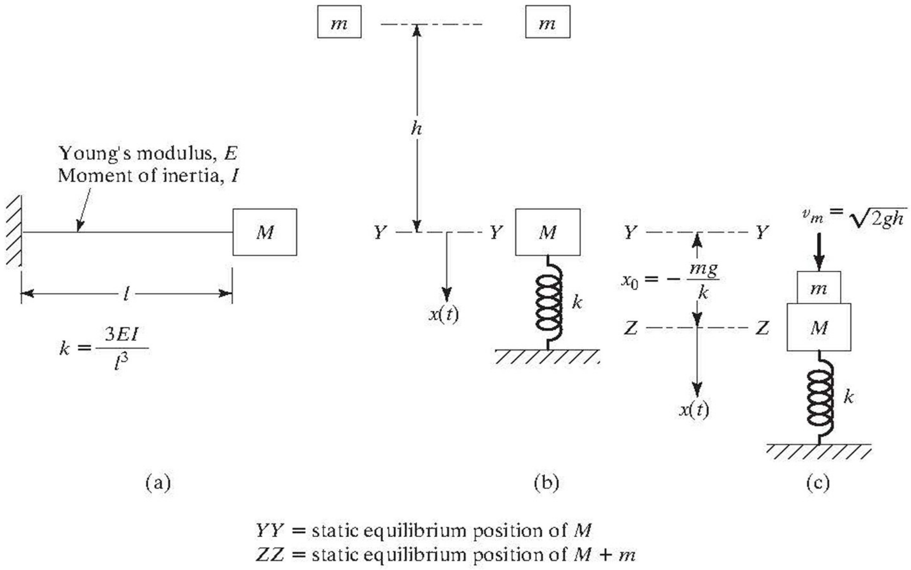


FIGURE 2.11 Response due to impact.

EXAMPLE 2.3

Young's Modulus from Natural Frequency Measurement

A simply supported beam of square cross section 5 mm $\times$ 5 mm and length 1 *m*, carrying a mass of 2.3 kg at the middle, is found to have a natural frequency of transverse vibration of 30 rad/s. Determine the Young's modulus of elasticity of the beam.

EXAMPLE 2.4 Natural Frequency of Cockpit of a Firetruck

The cockpit of a firetruck is located at the end of a telescoping boom, as shown in Fig. 2.12(a). The cockpit, along with the fireman, weighs 2000 N. Find the cockpit's natural frequency of vibration in the vertical direction.

Data: Young's modulus of the material: $E = 2.1 \times 10^{11}$ N/m^2; lengths: $l_1 = l_2 = l_3 = 3$ m; cross-sectional areas: $A_1 = 20$ cm^2, $A_2 = 10$ cm^2, $A_3 = 5$ cm^2.

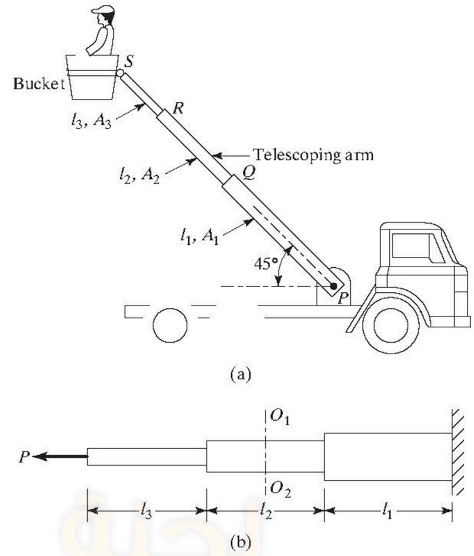


FIGURE 2.12 Telescoping boom of a fire truck.

EXAMPLE 2.10 Response of Anvil of a Forging Hammer

The anvil of a forging hammer weighs 5,000 N and is mounted on a foundation that has a stiffness of 5×10^6 N/m and a viscous damping constant of 10,000 N-s/m. During a particular forging operation, the tup (i.e., the falling weight or the hammer), weighing 1,000 N, is made to fall from a height of 2 m onto the anvil (Fig. 2.29(a)). If the anvil is at rest before impact by the tup, determine the response of the anvil after the impact. Assume that the coefficient of restitution between the anvil and the tup is 0.4.

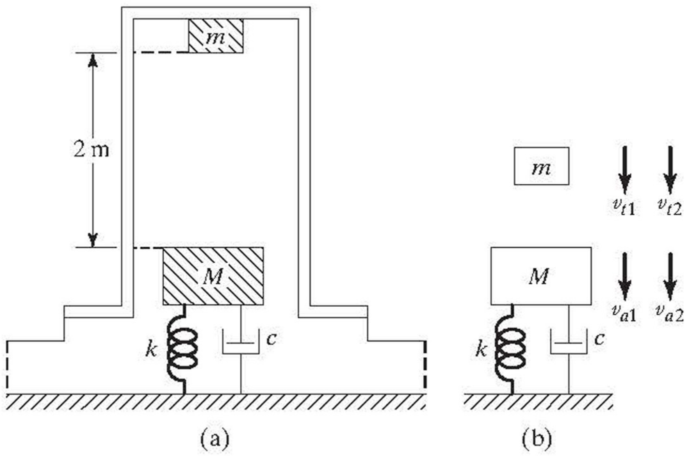


FIGURE 2.29 Forging hammer.

EXAMPLE 2.11 Shock Absorber for a Motorcycle

An underdamped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig. 2.30(a)). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig. 2.30(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 s and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

Approach: We use the equation for the logarithmic decrement in terms of the damping ratio, equation for the damped period of vibration, time corresponding to maximum displacement for an underdamped system, and envelope passing through the maximum points of an underdamped system.

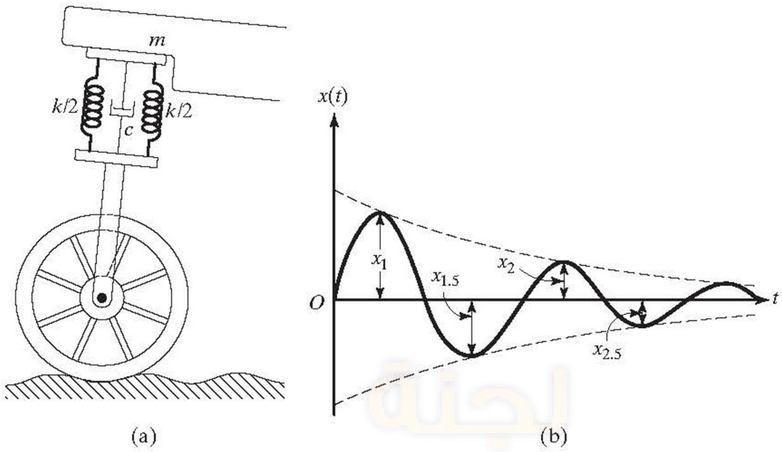


FIGURE 2.30 Shock absorber of a motorcycle.

EXAMPLE 2.14 Coefficient of Friction from Measured Positions of Mass

A metal block, placed on a rough surface, is attached to a spring and is given an initial displacement of 10 cm from its equilibrium position. After five cycles of oscillation in 2 s, the final position of the metal block is found to be 1 cm from its equilibrium position. Find the coefficient of friction between the surface and the metal block.

EXAMPLE 2.15 Pulley Subjected to Coulomb Damping

A steel shaft of length 1 m and diameter 50 mm is fixed at one end and carries a pulley of mass moment of inertia 25 kg-m^2 at the other end. A band brake exerts a constant frictional torque of 400 N-m around the circumference of the pulley. If the pulley is displaced by 6° and released, determine (1) the number of cycles before the pulley comes to rest and (2) the final settling position of the pulley.

2.13 Find the natural frequency of the pulley system shown in Fig. 2.56 by neglecting the friction and the masses of the pulleys.

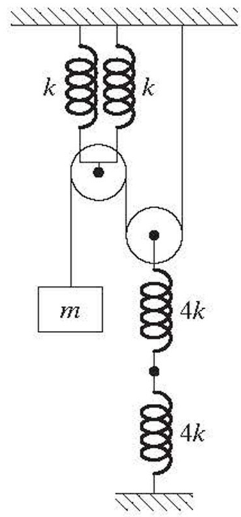


FIGURE 2.56

2.45–2.46 Draw the free-body diagram and derive the equation of motion using Newton's second law of motion for each of the systems shown in Figs. 2.85 and 2.86.

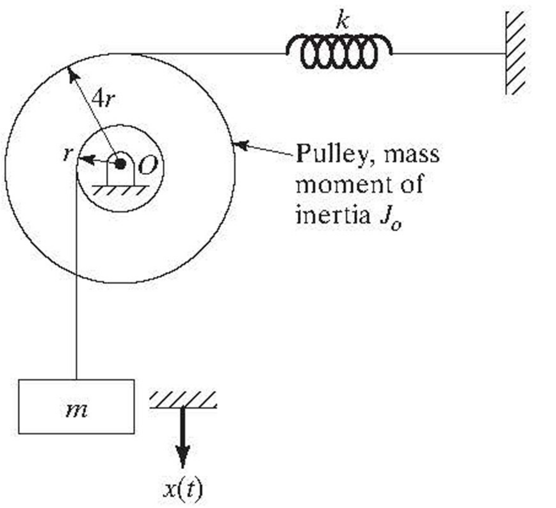


FIGURE 2.85

2.45–2.46 Draw the free-body diagram and derive the equation of motion using Newton's second law of motion for each of the systems shown in Figs. 2.85 and 2.86.

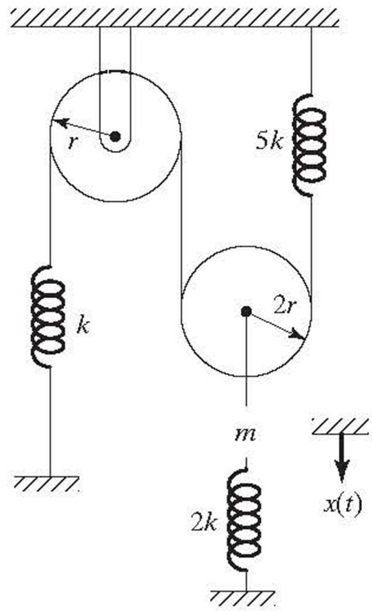


FIGURE 2.86

2.76 Find the equation of motion of the uniform rigid bar OA of length l and mass m shown in Fig. 2.98. Also find its natural frequency.

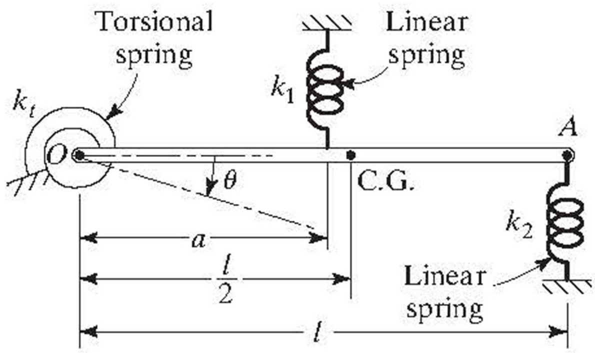


FIGURE 2.98

2.78 Derive the equation of motion of the system shown in Fig. 2.100, using the following methods: (a) Newton's second law of motion, (b) D'Alembert's principle, and (c) principle of virtual work.

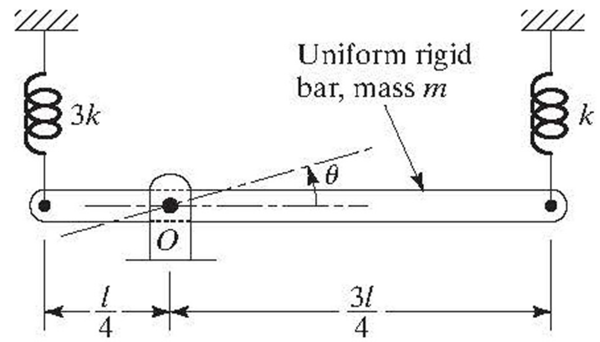


FIGURE 2.100

2.112–2.114 Derive the equation of motion and find the natural frequency of vibration of each of the systems shown in Figs. 2.110 to 2.112.

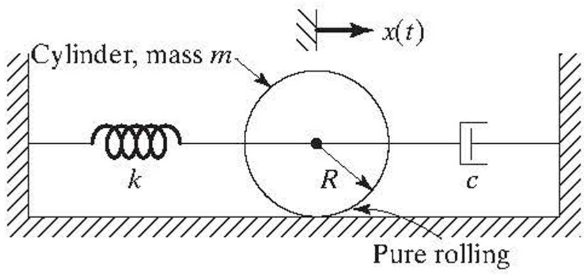


FIGURE 2.110

2.112–2.114 Derive the equation of motion and find the natural frequency of vibration of each of the systems shown in Figs. 2.110 to 2.112.

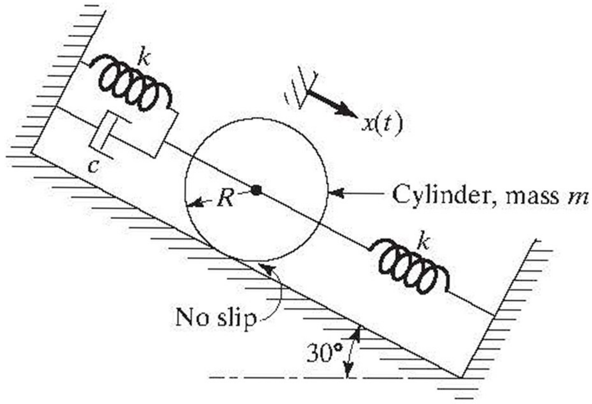


FIGURE 2.111

2.112–2.114 Derive the equation of motion and find the natural frequency of vibration of each of the systems shown in Figs. 2.110 to 2.112.

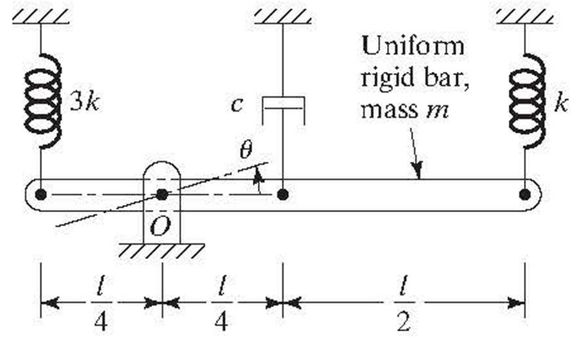


FIGURE 2.112

2.119 The system shown in Fig. 2.113 has a natural frequency of 5 Hz for the following data: $m = 10$ kg, $J_0 = 5$ kg-m^2, $r_1 = 10$ cm, $r_2 = 25$ cm. When the system is disturbed by giving it an initial displacement, the amplitude of free vibration is reduced by 80 percent in 10 cycles. Determine the values of k and c.

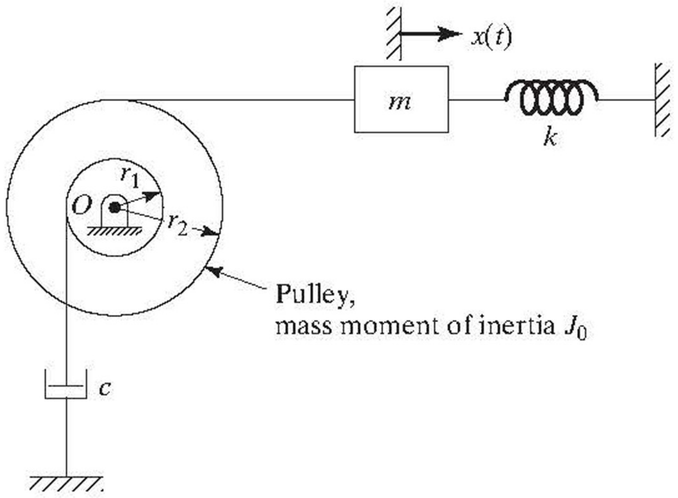


FIGURE 2.113

EXAMPLE 3.3 Total Response of a System

Find the total response of a single-degree-of-freedom system with $m = 10$ kg, $c = 20$ N-s/m, $k = 4000$ N/m, $x_0 = 0.01$ m, and $\dot{x}_0 = 0$ under the following conditions:

a. An external force $F(t) = F_0 \cos \omega t$ acts on the system with $F_0 = 100$ N and $\omega = 10$ rad/s.
b. Free vibration with $F(t) = 0$.

Solution:

a. From the given data, we obtain

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025 \text{ m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{(4000)(10)}} = 0.05$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.05)^2}(20) = 19.974984 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.025}{\left[(1-0.05^2)^2 + (2\cdot 0.5 \cdot 0.5)^2\right]^{1/2}} = 0.03326 \text{ m} \quad \text{(E.1)}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2\cdot 0.05 \cdot 0.5}{1-0.5^2}\right) = 3.814075° \quad \text{(E.2)}$$

Using the initial conditions, $x_0 = 0.01$ and $\dot{x}_0 = 0$, Eq. (3.36) yields:

$$0.01 = X_0 \cos \phi_0 + (0.03326)(0.997785)$$

or

$$X_0 \cos \phi_0 = -0.023186 \quad \text{(E.3)}$$

$$0 = -(0.05)(20)\, X_0 \cos \phi_0 + X_0(19.974984) \sin \phi_0 + (0.03326)(10) \sin(3.814075°) \quad \text{(E.4)}$$

Substituting the value of $X_0 \cos \phi_0$ from Eq. (E.3) into (E.4), we obtain

$$X_0 \sin \phi_0 = -0.002268 \quad \text{(E.5)}$$

Solution of Eqs. (E.3) and (E.5) yields

$$X_0 = \left[(X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2\right]^{1/2} = 0.023297 \quad \text{(E.6)}$$

and

$$\tan \phi_0 = \frac{X_0 \sin \phi_0}{X_0 \cos \phi_0} = 0.0978176$$

or

$$\phi_0 = 5.586765° \quad \text{(E.7)}$$

b. For free vibration, the total response is given by

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) \quad \text{(E.8)}$$

Using the initial conditions $x(0) = x_0 = 0.01$ and $\dot{x}(0) = \dot{x}_0 = 0$, X_0 and ϕ_0 of Eq. (E.8) can be determined as (see Eqs. 2.73 and 2.75):

$$X_0 = \left[x_0^2 + \left(\frac{\zeta \omega_n x_0}{\omega_d}\right)^2\right]^{1/2} = \left[0.01^2 + \left(\frac{0.05 \cdot 20 \cdot 0.01}{19.974984}\right)^2\right]^{1/2} = 0.010012 \quad \text{(E.9)}$$

$$\phi_0 = \tan^{-1}\left(-\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d\, x_0}\right) = \tan^{-1}\left(-\frac{0.05 \cdot 20}{19.974984}\right) = -2.865984° \quad \text{(E.10)}$$

Note that the constants X_0 and ϕ_0 in cases (a) and (b) are very different.

EXAMPLE 3.4

Vehicle Moving on a Rough Road

Figure 3.18 shows a simple model of a motor vehicle that can vibrate in the vertical direction while traveling over a rough road. The vehicle has a mass of 1200 kg. The suspension system has a spring constant of 400 kN/m and a damping ratio of $\zeta = 0.5$. If the vehicle speed is 20 km/hr, determine the displacement amplitude of the vehicle. The road surface varies sinusoidally with an amplitude of $Y = 0.05$ m and a wavelength of 6 m.

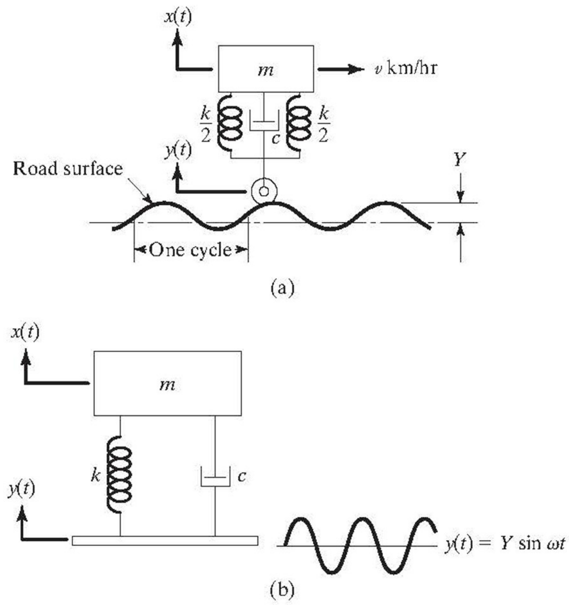


FIGURE 3.18 Vehicle moving over a rough road.

Solution: The frequency ω of the base excitation can be found by dividing the vehicle speed v km/hr by the length of one cycle of road roughness:

$$\omega = 2\pi f = 2\pi\left(\frac{v \times 1000}{3600}\right)\frac{1}{6} = 0.290889v \text{ rad/s}$$

For $v = 20$ km/hr, $\omega = 5.81778$ rad/s. The natural frequency of the vehicle is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{400 \times 10^3}{1200}\right)^{1/2} = 18.2574 \text{ rad/s}$$

and hence the frequency ratio r is

$$r = \frac{\omega}{\omega_n} = \frac{5.81778}{18.2574} = 0.318653$$

The amplitude ratio can be found from Eq. (3.68):

$$\frac{X}{Y} = \left\{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right\}^{1/2} = \left\{\frac{1 + (2 \times 0.5 \times 0.318653)^2}{(1 - 0.318653)^2 + (2 \times 0.5 \times 0.318653)^2}\right\}^{1/2}$$

$$= 1.100964$$

Thus the displacement amplitude of the vehicle is given by

$$X = 1.100964Y = 1.100964(0.05) = 0.055048 \text{ m}$$

This indicates that a 5-cm bump in the road is transmitted as a 5.5-cm bump to the chassis and the passengers of the car. Thus in the present case the passengers feel an amplified motion (see Problem 3.107 for other situations).

EXAMPLE 3.5 Machine on Resilient Foundation

A heavy machine, weighing 3000 N, is supported on a resilient foundation. The static deflection of the foundation due to the weight of the machine is found to be 7.5 cm. It is observed that the machine vibrates with an amplitude of 1 cm when the base of the foundation is subjected to harmonic oscillation at the undamped natural frequency of the system with an amplitude of 0.25 cm. Find

a. the damping constant of the foundation,
b. the dynamic force amplitude on the base, and
c. the amplitude of the displacement of the machine relative to the base.

Solution:

a. The stiffness of the foundation can be found from its static deflection: k = weight of machine$/\delta_{st} = 3000/0.075 = 40{,}000$ N/m.

At resonance ($\omega = \omega_n$ or $r = 1$), Eq. (3.68) gives

$$\frac{X}{Y} = \frac{0.010}{0.0025} = 4 = \left[\frac{1 + (2\zeta)^2}{(2\zeta)^2}\right]^{1/2} \tag{E.1}$$

The solution of Eq. (E.1) gives $\zeta = 0.1291$. The damping constant is given by

$$\begin{aligned} c = \zeta \cdot c_c = \zeta 2\sqrt{km} &= 0.1291 \times 2 \times \sqrt{40{,}000 \times (3000/9.81)} \\ &= 903.0512 \text{ N-s/m} \end{aligned} \tag{E.2}$$

b. The dynamic force amplitude on the base at $r = 1$ can be found from Eq. (3.74):

$$F_T = Yk\left[\frac{1 + 4\zeta^2}{4\zeta^2}\right]^{1/2} = kX = 40{,}000 \times 0.01 = 400 \text{ N} \tag{E.3}$$

c. The amplitude of the relative displacement of the machine at $r = 1$ can be obtained from Eq. (3.77):

$$Z = \frac{Y}{2\zeta} = \frac{0.0025}{2 \times 0.1291} = 0.00968 \text{ m} \tag{E.4}$$

It can be noticed that $X = 0.01$ m, $Y = 0.0025$ m, and $Z = 0.00968$ m; therefore, $Z \neq X - Y$. This is due to the phase differences between x, y, and z.

EXAMPLE 3.6 Deflection of an Electric Motor due to Rotating Unbalance

An electric motor of mass M, mounted on an elastic foundation, is found to vibrate with a deflection of 0.15 m at resonance (Fig. 3.20). It is known that the unbalanced mass of the motor is 8% of the mass of the rotor due to manufacturing tolerances used, and the damping ratio of the foundation is $\zeta = 0.025$. Determine the following:

a. the eccentricity or radial location of the unbalanced mass (e),
b. the peak deflection of the motor when the frequency ratio varies from resonance, and
c. the additional mass to be added uniformly to the motor if the deflection of the motor at resonance is to be reduced to 0.1 m.

Assume that the eccentric mass remains unaltered when the additional mass is added to the motor.

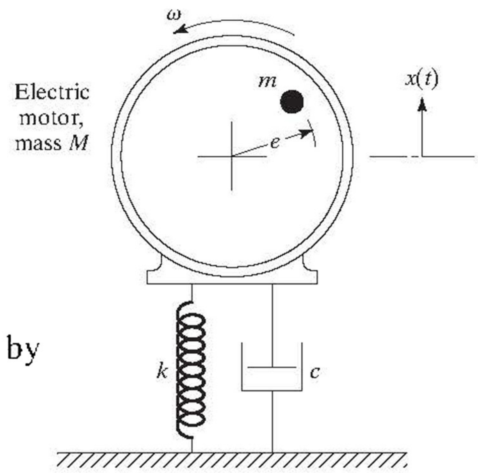


FIGURE 3.20

Solution

a. From Eq. (3.81), the deflection at resonance ($r = 1$) is given by

$$\frac{MX}{me} = \frac{1}{2\zeta} = \frac{1}{2(0.025)} = 20$$

from which the eccentricity can be found as

$$e = \frac{MX}{20m} = \frac{M(0.15)}{20(0.08\,M)} = 0.09375 \text{ m}$$

b. The peak deflection of the motor is given by Eq. (3.83):

$$\left(\frac{MX}{me}\right)_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{2(0.025)\sqrt{1-0.025^2}} = 20.0063$$

from which the peak deflection can be determined as

$$X_{\max} = \frac{20.0063me}{M} = \frac{20.0063(0.08M)(0.09375)}{M} = 0.150047 \text{ m}$$

c. If the additional mass added to the motor is denoted as M_a, the corresponding deflection is given by Eq. (3.81):

$$\frac{(M + M_a)(0.1)}{(0.08M)(0.09375)} = 20$$

which yields $M_a = 0.5M$. Thus the mass of the motor is to be increased by 50% in order to reduce the deflection at resonance from 0.15 m to 0.10 m.

EXAMPLE 3.8 Spring-Mass System with Coulomb Damping

A spring-mass system, having a mass of 10 kg and a spring of stiffness of 4000 N/m, vibrates on a horizontal surface. The coefficient of friction is 0.12. When subjected to a harmonic force of frequency 2 Hz, the mass is found to vibrate with an amplitude of 40 mm. Find the amplitude of the harmonic force applied to the mass.

Solution: The vertical force (weight) of the mass is $N = mg = 10 \times 9.81 = 98.1$ N. The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

and the frequency ratio is

$$\frac{\omega}{\omega_n} = \frac{2 \times 2\pi}{20} = 0.6283$$

The amplitude of vibration X is given by Eq. (3.93):

$$X = \frac{F_0}{k}\left[\frac{1 - \left(\dfrac{4\mu N}{\pi F_0}\right)^2}{\left\{1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right\}^2}\right]^{1/2}$$

$$0.04 = \frac{F_0}{4000}\left[\frac{1 - \left\{\dfrac{4(0.12)(98.1)}{\pi F_0}\right\}^2}{(1 - 0.6283^2)^2}\right]^{1/2}$$

The solution of this equation gives $F_0 = 97.9874$ N.

Section 3.3 Response of an Undamped System Under Harmonic Force

3.1 A weight of 50 N is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force of amplitude 60 N and frequency 6 Hz. Find (a) the extension of the spring due to the suspended weight, (b) the static displacement of the spring due to the maximum applied force, and (c) the amplitude of forced motion of the weight.

(a) $\delta = \frac{W}{k} = \frac{50}{4000} = 0.0125 \text{ m}$

(b) $\delta_{st} = \frac{F_0}{k} = \frac{60}{4000} = 0.015 \text{ m}$

(c) $\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{4000 \times 9.81}{50}\right)^{1/2} = 28.0143 \text{ rad/sec}$

$\omega = 6 \text{ Hz} = 37.6992 \text{ rad/sec}$

$$X = \delta_{st} \left| \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \right| = 0.015 \left| \frac{1}{1-\left(\frac{37.6992}{28.0143}\right)^2} \right| = 0.0185 \text{ m}$$

3.2 A spring-mass system is subjected to a harmonic force whose frequency is close to the natural frequency of the system. If the forcing frequency is 39.8 Hz and the natural frequency is 40.0 Hz, determine the period of beating.

$$\tau_b = \frac{2\pi}{\omega_n - \omega} = \frac{2\pi}{2\pi\,(40.0 - 39.8)} = 5 \text{ sec}$$

3.3 Consider a spring-mass system, with $k = 4000$ N/m and $m = 10$ kg, subject to a harmonic force $F(t) = 400 \cos 10t$ N. Find and plot the total response of the system under the following initial conditions:

a. $x_0 = 0.1$ m, $\dot{x}_0 = 0$
b. $x_0 = 0$, $\dot{x}_0 = 10$ m/s
c. $x_0 = 0.1$ m, $\dot{x}_0 = 10$ m/s

$k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 10t$ N

$F_0 = 400$ N, $\omega = 10$ rad/s

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\frac{\omega}{\omega_n} = \frac{10}{20} = 0.5 < 1$

Response is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) becomes

$$x(t) = \left\{ 0.1 - \frac{400}{4000 - 10(100)} \right\} \cos 20t + \frac{400}{4000 - 10(100)} \cos 10t$$

$$= -0.033333 \cos 20t + 0.133333 \cos 10t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) becomes

$$x(t) = \left\{ 0 - \frac{400}{4000 - 10(100)} \right\} \cos 20t + \frac{10}{20} \sin 20t + \left\{ \frac{400}{4000 - 10(100)} \right\} \cos 10t$$

$$= -0.133333 \cos 20t + 0.5 \sin 20t + 0.133333 \cos 10t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) becomes

$$x(t) = \left\{ 0.1 - \frac{400}{4000 - 10(100)} \right\} \cos 20t + \frac{10}{20} \sin 20t + \left\{ \frac{400}{4000 - 10(100)} \right\} \cos 10t$$

$$= -0.033333 \cos 20t + 0.5 \sin 20t + 0.133333 \cos 10t \qquad (E.4)$$

3.4 Consider a spring-mass system, with $k = 4000$ N/m and $m = 10$ kg, subject to a harmonic force $F(t) = 400 \cos 20t$ N. Find and plot the total response of the system under the following initial conditions:

a. $x_0 = 0.1$ m, $\dot{x}_0 = 0$
b. $x_0 = 0$, $\dot{x}_0 = 10$ m/s
c. $x_0 = 0.1$ m, $\dot{x}_0 = 10$ m/s

$k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 20t$ N,
$F_0 = 400$ N, $\omega = 20$ rad/s

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}, \quad \frac{\omega}{\omega_n} = \frac{20}{20} = 1$$

Response is given by Eq. (3.15):

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st}\, \omega_n t}{2} \sin \omega_n t \qquad (E.1)$$

where $\delta_{st} = F_0/k = 400/4000 = 0.1$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:
Eq. (E.1) gives

$$x(t) = 0.1 \cos 20t + \frac{(0.1)(20)t}{2} \sin 20t$$

$$= 0.1 \cos 20t + t \sin 20t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:
Eq. (E.1) gives

$$x(t) = \frac{10}{20} \sin 20t + \frac{(0.1)(20)t}{2} \sin 20t$$

$$= 0.5 \sin 20t + t \sin 20t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:
Eq. (E.1) gives

$$x(t) = 0.1 \cos 20t + \frac{10}{20} \sin 20t + \frac{(0.1)(20)t}{2} \sin 20t$$

$$= 0.1 \cos 20t + 0.5 \sin 20t + t \sin 20t \qquad (E.4)$$

3.5 Consider a spring-mass system, with $k = 4000$ N/m and $m = 10$ kg, subject to a harmonic force $F(t) = 400 \cos 20.1t$ N. Find and plot the total response of the system under the following initial conditions:

a. $x_0 = 0.1$ m, $\dot{x}_0 = 0$
b. $x_0 = 0$, $\dot{x}_0 = 10$ m/s
c. $x_0 = 0.1$ m, $\dot{x}_0 = 10$ m/s

$k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 20.1\,t$ N

$F_0 = 400$ N, $\omega = 20.1$ rad/s, $\omega^2 = 404.01$ (rad/s)^2

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s

Solution is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2} \right) \cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) reduces to

$$x(t) = \left\{ 0.1 - \frac{400}{4000 - 10\,(404.01)} \right\} \cos 20\,t + \left\{ \frac{400}{4000 - 10\,(404.01)} \right\} \cos 20.1\,t$$

$$= 10.075062 \cos 20t - 9.975062 \cos 20.1\,t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) reduces to

$$x(t) = -\left\{ \frac{400}{4000 - 10\,(404.01)} \right\} \cos 20\,t + \frac{10}{20} \sin 20\,t + \left\{ \frac{400}{4000 - 10\,(404.01)} \right\} \cos 20.1\,t$$

$$= 9.975062 \cos 20\,t + 0.5 \sin 20\,t - 9.975062 \cos 20.1\,t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = \left\{ 0.1 - \frac{400}{4000 - 10\,(404.01)} \right\} \cos 20\,t + \frac{10}{20} \sin 20\,t + \left\{ \frac{400}{4000 - 10\,(404.01)} \right\} \cos 20.1\,t$$

$$= 10.075062 \cos 20\,t + 0.5 \sin 20\,t - 9.975062 \cos 20.1\,t \qquad (E.4)$$

3.9 A spring-mass system with $m = 10$ kg and $k = 5000$ N/m is subjected to a harmonic force of amplitude 250 N and frequency ω. If the maximum amplitude of the mass is observed to be 100 mm, find the value of ω.

$$\omega_n = \sqrt{k/m} = \sqrt{5000/10} = 22.3607 \text{ rad/sec}$$

$$\delta_{st} = F_0/k = 250/5000 = 0.05 \text{ m}$$

$$X = \delta_{st} \left\{ \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right\}$$

i.e., $$\omega = \omega_n \left(1 - \frac{\delta_{st}}{X}\right)^{\frac{1}{2}} = 22.3607 \left[1 - \frac{0.05}{0.10}\right]^{\frac{1}{2}}$$

$$= 15.8114 \text{ rad/sec}$$

3.24 Derive the equation of motion and find the steady-state response of the system shown in Fig. 3.44 for rotational motion about the hinge O for the following data: $k_1 = k_2 = 5000$ N/m, $a = 0.25$ m, $b = 0.5$ m, $l = 1$ m, $M = 50$ kg, $m = 10$ kg, $F_0 = 500$ N, $\omega = 1000$ rpm.

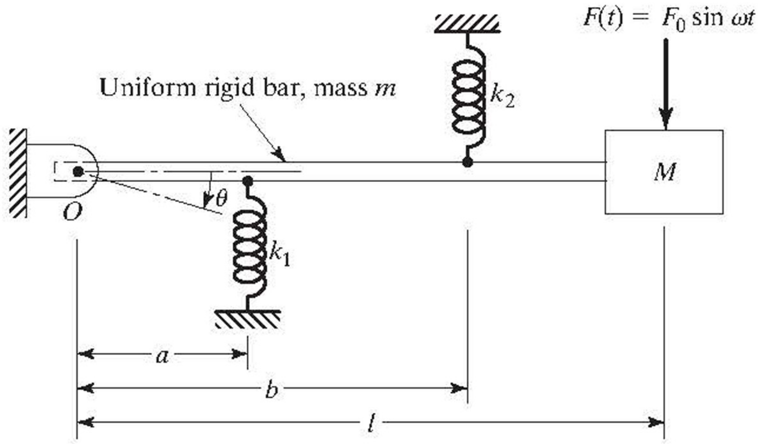
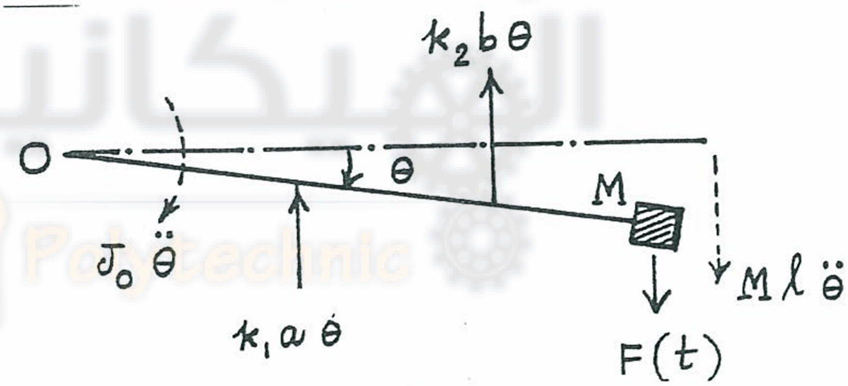


FIGURE 3.44



Equation of motion for rotational motion about the hinge O:

$$(J_0 + M\,\ell^2)\,\ddot{\theta} + (k_1\,a^2 + k_2\,b^2)\,\theta = F(t)\,\ell = F_0\,\ell\,\sin\omega t \tag{1}$$

Steady state response (using Eqs. (3.3) and (3.6)):

$$\theta_p(t) = \Theta\,\sin\omega t \tag{2}$$

$$\text{where}\quad \Theta = \frac{F_0\,\ell}{(k_1\,a^2 + k_2\,b^2) - (J_0 + M\,\ell^2)\,\omega^2} \tag{3}$$

$$\text{and}\quad J_0 = \frac{m\,\ell^2}{12} + m\,(\frac{\ell}{2})^2 = \frac{1}{3}\,m\,\ell^2 \tag{4}$$

For given data, $J_0 = \frac{1}{3}\,(10)\,(1^2) = 3.3333$ kg–m^2, $\omega = \frac{1000\,(2\,\pi)}{60} = 104.72$ rad/sec,

and

$$\Theta = \frac{500\,(1)}{5000\,(0.25^2 + 0.5^2) - (3.3333 + 50\,(1^2))\,(104.72^2)} = -\,8.5718\,(10^{-4})\text{ rad}$$

3.25 Derive the equation of motion and find the steady-state solution of the system shown in Fig. 3.45 for rotational motion about the hinge O for the following data: $k = 5000$ N/m, $l = 1$ m, $m = 10$ kg, $M_0 = 100$ N-m, $\omega = 1000$ rpm.

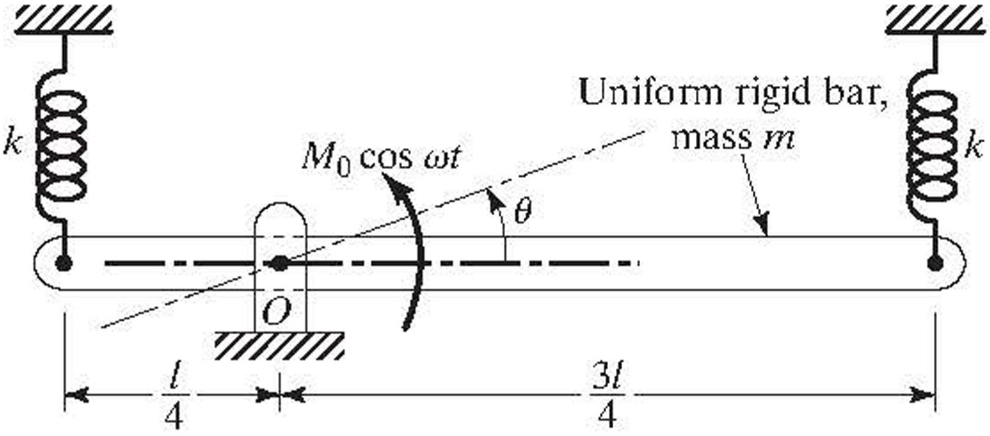


FIGURE 3.45

Equation of motion for rotation about O:

$$J_0 \ddot{\theta} = -k \frac{\theta \ell}{4} \frac{\ell}{4} - k \frac{\theta \, 3\ell}{4} \frac{3\ell}{4} + M_0 \cos \omega t$$

$$\text{i.e.,} \quad J_0 \ddot{\theta} + \left(\frac{5}{8} k \ell^2 \right) \theta = M_0 \cos \omega t$$

where $J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10)(1^2) = 1.4583$ kg–m^2

and $\omega = 1000$ rpm $= 104.72$ rad/sec. Steady state solution is:

$$\theta_p(t) = \Theta \cos \omega t$$

where

$$\Theta = \frac{M_0}{\frac{5}{8} k \ell^2 - J_0 \omega^2} = \frac{100}{5000 \left(\frac{5}{8}\right)(1^2) - 1.4583 \,(104.72^2)} = -0.007772 \text{ rad}$$

Section 3.4 Response of a Damped System under Harmonic Force

3.26 Consider a spring-mass-damper system with $k = 4000$ N/m, $m = 10$ kg, and $c = 40$ N-s/m. Find the steady-state and total responses of the system under the harmonic force $F(t) = 200 \cos 10t$ N and the initial conditions $x_0 = 0.1$ m and $\dot{x}_0 = 0$.

$k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200 \cos 10t$,

$F_0 = 200$ N, $\omega = 10$ rad/s, $x_0 = 0.1$ m, $\dot{x}_0 = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}, \quad \delta_{st} = \frac{F_0}{k} = \frac{200}{4000} = 0.05 \text{ m}$$

$$\zeta = c/c_c = \left(c/2\sqrt{km}\right) = \left(40/2\sqrt{4000(10)}\right) = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.1)^2}\,(20) = 19.899749 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

$$X = \delta_{st}/\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \frac{0.05}{\left\{(1-0.5^2)^2 + (2(0.1)(0.5))^2\right\}^{\frac{1}{2}}}$$

$$= 0.066082 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2 * 0.1 * 0.5}{1-0.5^2}\right) = 0.132552 \text{ rad}$$

steady state response, Eq. (3.25):

$$x_p(t) = X \cos(\omega t - \phi)$$

$$= 0.066082 \cos(10t - 0.132552) \text{ m}$$

Total response, Eq. (3.35):

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi) \qquad (E.1)$$

Using the initial conditions x_0 and $\dot{x}_0$, Eq. (E.1) gives

$$x_0 = X_0 \cos\phi_0 + X \cos\phi \qquad (E.2)$$

or $$X_0 \cos\phi_0 = x_0 - X\cos\phi \qquad (E.3)$$

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 + \omega_d X_0 \sin\phi_0 + \omega X \sin\phi \qquad (E.4)$$

or $$X_0 \sin\phi_0 = \frac{1}{\omega_d}\left\{\dot{x}_0 + \zeta\omega_n X_0 \cos\phi_0 - \omega X \sin\phi\right\} \qquad (E.5)$$

For known values, Eqs. (E.3) and (E.5) yield

$$X_0 \cos\phi_0 = 0.034498, \qquad X_0 \sin\phi_0 = -0.000922$$

Hence

$$X_0 = \left\{(X_0 \cos\phi_0)^2 + (X_0 \sin\phi_0)^2\right\}^{\frac{1}{2}} = 0.034510$$

$$\phi_0 = \tan^{-1}\left(\frac{X_0 \sin\phi_0}{X_0 \cos\phi_0}\right) = -0.026710$$

Thus the total response, Eq. (E.1), will be

$$x(t) = 0.034510\, e^{-2t} \cos(19.899749\, t + 0.026710)$$
$$+ 0.066082 \cos(10t - 0.132552) \text{ m} \qquad (E.6)$$

3.34 A spring-mass-damper system is subjected to a harmonic force. The amplitude is found to be 20 mm at resonance and 10 mm at a frequency 0.75 times the resonant frequency. Find the damping ratio of the system.

Eq. (3.34): $$\frac{X_{res}}{\delta_{st}} = \left.\frac{X}{\delta_{st}}\right|_{\omega=\omega_n} = \frac{1}{2\zeta}$$

i.e., $$\delta_{st} = 2\,\zeta\left(\frac{20}{1000}\right) = 0.04\,\zeta \qquad (E_1)$$

Eq. (3.30): $$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\ ; \qquad r = 0.75 = \frac{\omega}{\omega_n}$$

i.e., $$\frac{0.01}{\delta_{st}} = \frac{1}{\sqrt{(1-0.75^2)^2+(2\,\zeta\times 0.75)^2}} \qquad (E_2)$$

Eqs. (E_1) and (E_2) give

$$\frac{0.01}{0.04\,\zeta} = \frac{1}{\sqrt{0.1914+2.25\,\zeta^2}}$$

i.e., $$0.1914 + 2.25\,\zeta^2 = 16\,\zeta^2$$

i.e., $$\zeta = 0.1180$$

3.41 A torsional system consists of a disc of mass moment of inertia $J_0 = 10$ kg-m^2, a torsional damper of damping constant $c_t = 300$ N-m-s/rad, and a steel shaft of diameter 4 cm and length 1 m (fixed at one end and attached to the disc at the other end). A steady angular oscillation of amplitude 2° is observed when a harmonic torque of magnitude 1000 N-m is applied to the disc. (a) Find the frequency of the applied torque, and (b) find the maximum torque transmitted to the support.

$$k_t = \frac{\pi G}{32\, l} d^4 = \frac{\pi\,(79.3 \times 10^9)}{32\,(1)} \left(\frac{4}{100}\right)^4 = 19930.31 \text{ N-m/rad}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{19930.31}{10}} = 44.6434 \text{ rad/sec}$$

$$\theta_{st} = M_{t0}/k_t = 1000/19930.31 = 0.0502 \text{ rad}$$

$$\zeta_t = \frac{c_t}{2 J_0 \omega_n} = \frac{300}{2(10)(44.6434)} = 0.336$$

(a) Eq. (3.30), when written for a torsional system, gives

$$\frac{\Theta}{\theta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

i.e., $$\frac{(2/57.2956)}{0.0502} = \frac{1}{\sqrt{(1-r^2)^2 + (2 \times 0.336\, r)^2}}$$

i.e., $$r^4 - 1.5484\, r^2 - 1.0679 = 0$$

i.e., $$r^2 = 2.0655,\ -0.5171$$

$$\therefore\ \omega = r\,\omega_n = \sqrt{2.0655}\,(44.6434) = 64.16 \text{ rad/sec}$$

(b) Maximum torque transmitted to the support:

$$M_t(t) = k_t\,\theta(t) + c_t\,\dot\theta(t)$$
$$= k_t\,\Theta\,\cos(\omega t - \phi) - c_t\,\Theta\,\omega\,\sin(\omega t - \phi)$$

$$(M_t)_{max} = \sqrt{(k_t\,\Theta)^2 + (c_t\,\Theta\,\omega)^2}$$
$$= \sqrt{\left\{19930.31\left(\frac{2}{57.2956}\right)\right\}^2 + \left\{300\left(\frac{2}{57.2956}\right)(64.16)\right\}^2}$$

$$= 967.2 \text{ N-m}$$

3.42 For a vibrating system, $m = 10$ kg, $k = 2500$ N/m, and $c = 45$ N-s/m. A harmonic force of amplitude 180 N and frequency 3.5 Hz acts on the mass. If the initial displacement and velocity of the mass are 15 mm and 5 m/s, find the complete solution representing the motion of the mass.

Complete solution is $x(t) = X_0 \, e^{-\zeta \omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi)$

$\omega = 2\pi(3.5) = 21.9912$ rad/sec, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{10}} = 15.8114$ rad/sec

$\delta_{st} = \frac{F_0}{k} = \frac{180}{2500} = 0.072$ m

$\zeta = \frac{c}{2 m \omega_n} = \frac{45}{2(10)(15.8114)} = 0.1423$, $r = \frac{\omega}{\omega_n} = \frac{21.9912}{15.8114} = 1.3908$

$\zeta \omega_n = 2.25$, $\omega_d = \sqrt{1-\zeta^2}\, \omega_n = 15.6505$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.072}{\left[(1-1.3908^2)^2 + (2 \times 0.1423 \times 1.3908)^2\right]^{1/2}}$$

$= 0.07095$ m

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.3958}{-0.9343}\right) = -22.9591^\circ$$

$$x(t) = X_0 \, e^{-2.25 t} \cos(15.6505 t + \phi_0) + 0.07095 \cos(21.9912 t + 22.9591^\circ)$$

$$\dot{x}(t) = -2.25 X_0 \, e^{-2.25 t} \cos(15.6505 t + \phi_0) - 15.6505 X_0 \, e^{-2.25 t} \sin(15.6505 t + \phi_0)$$
$$- 21.9912 (0.07095) \sin(21.9912 t + 22.9591^\circ)$$

$x(0) = 0.015 = X \cos\phi_0 + 0.07095 \cos 22.9591^\circ$

$X \cos\phi_0 = -0.05033$ ---- (E_1)

$\dot{x}(0) = 5 = -2.25 X_0 \cos\phi_0 - 15.6505 X_0 \sin\phi_0 - 1.5603 \sin 22.9591^\circ$

$$X \sin\phi_0 = \frac{-0.6086 - 2.25 X_0 \cos\phi_0 - 5}{15.6505} = -0.3511 \quad ---- (E_2)$$

Eqs. (E_1) and (E_2) give

$X_0 = \left\{(-0.05033)^2 + (-0.3511)^2\right\}^{1/2} = 0.3547$

$\phi_0 = \tan^{-1}\left(\frac{0.3511}{0.05033}\right) = \tan^{-1}(6.9760) = 81.8423^\circ$

3.44 The landing gear of an airplane can be idealized as the spring-mass-damper system shown in Fig. 3.52. If the runway surface is described $y(t) = y_0 \cos \omega t$, determine the values of k and c that limit the amplitude of vibration of the airplane (x) to 0.1 m. Assume $m = 2000$ kg, $y_0 = 0.2$ m, and $\omega = 157.08$ rad/s.

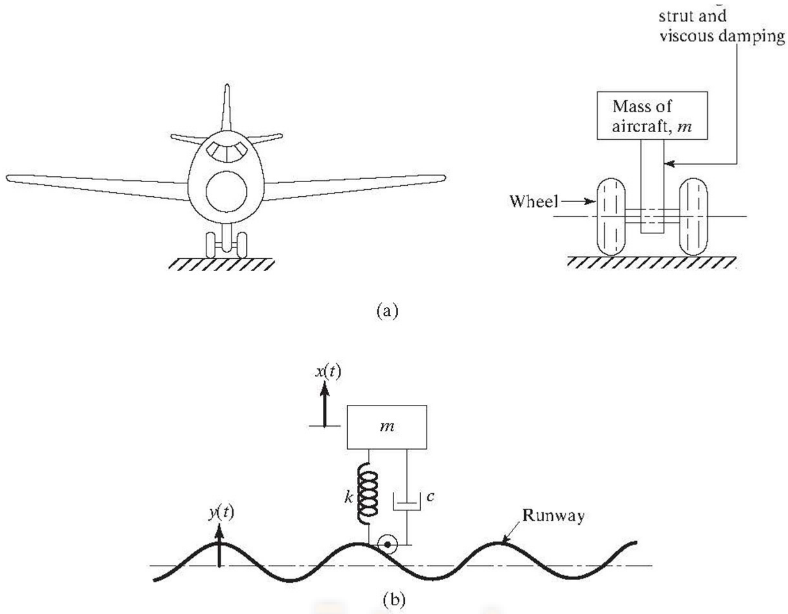


FIGURE 3.52 Modeling of landing gear.

Amplitude of vibration under base excitation:

$$X = Y \left\{ \frac{\sqrt{k^2 + (c\,\omega)^2}}{\left[\left(k - m\,\omega^2\right)^2 + (c\,\omega)^2\right]^{\frac{1}{2}}} \right\}$$

$$= \frac{(0.2)\sqrt{k^2 + c^2\,(157.08)^2}}{\left[\left\{k - 2000\,(157.08)^2\right\}^2 + c^2\,(157.08)^2\right]^{\frac{1}{2}}} = 0.1 \text{ m} \qquad (1)$$

Let $k = 5\,(10^6)$ N/m. Then Eq. (1) gives:

$$\frac{\sqrt{25\,(10^{12}) + 2.4674\,(10^4)\,c^2}}{\sqrt{1966.7717\,(10^{12}) + 2.4674\,(10^4)\,c^2}} = 0.5$$

i.e., $1.85055\,(10^4)\,c^2 = 466.6929\,(10^{12})$ i.e., $c = 158805.0$ N–s/m

3.46 Derive the equation of motion and find the steady-state response of the system shown in Fig. 3.54 for rotational motion about the hinge O for the following data: $k = 5000$ N/m, $l = 1$ m, $c = 1000$ N-s/m, $m = 10$ kg, $M_0 = 100$ N-m, $\omega = 1000$ rpm.

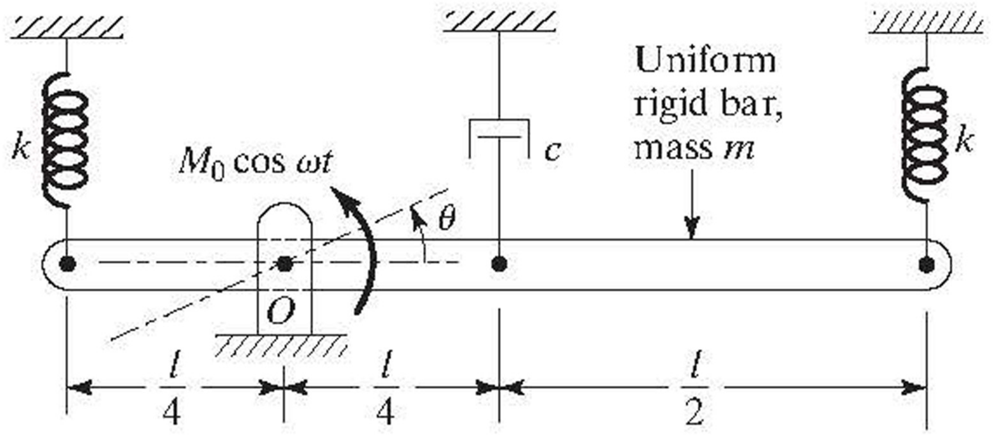


FIGURE 3.54

Equation of motion:

$$I_0 \ddot{\theta} + \left(k \frac{\ell}{4} \theta\right) \frac{\ell}{4} + \left(c \frac{\ell}{4} \dot{\theta}\right) \frac{\ell}{4} + \left(k \frac{3\ell}{4} \theta\right) \frac{3\ell}{4} = M_0 \cos \omega t$$

$$\text{or} \quad I_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{5}{8} k \ell^2 \theta = M_0 \cos \omega t$$

$$\text{where } I_0 = \frac{m \ell^2}{12} + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10)(1^2) = 1.4583 \text{ kg—m}^2$$

$$\frac{c \ell^2}{16} = \frac{(1000)(1^2)}{16} = 62.5 \text{ N—m—s/rad}$$

$$\frac{5}{8} k \ell^2 = \frac{5}{8} (5000)(1^2) = 3125.0 \text{ N—m/rad}$$

$$\omega = \frac{1000 (2\pi)}{60} = 104.72 \text{ rad/sec}$$

Equation of motion becomes:

$$1.4583 \ddot{\theta} + 62.5 \dot{\theta} + 3125.0 \theta = 100 \cos 104.72 t$$

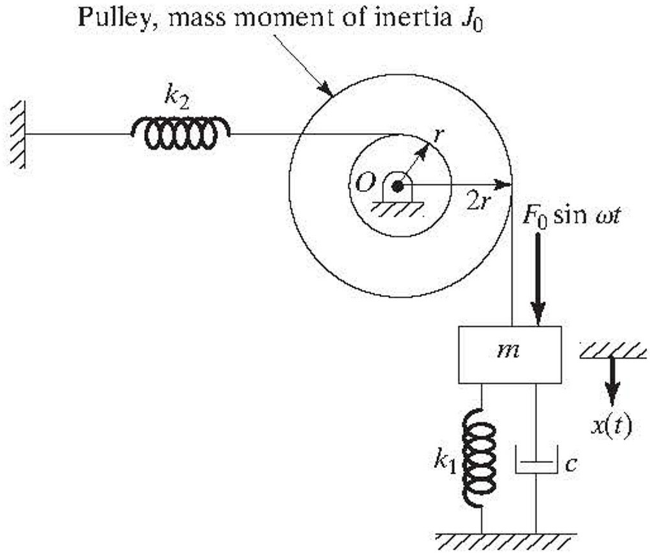
Steady state response is given by Eq. (3.28):

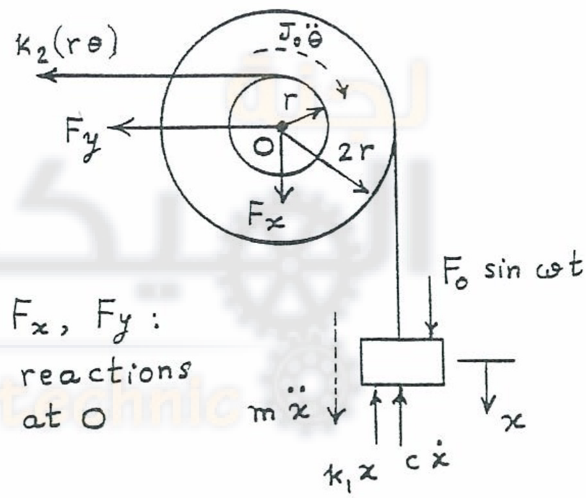
$$\theta_p(t) = \Theta \cos(\omega t - \phi) = \Theta \cos(104.72 t - \phi).sp$$

$$\text{where} \quad \Theta = \frac{100}{\left[\left\{3125.0 - 1.4583 (104.72^2)\right\}^2 + \left\{62.5 (104.72)\right\}^2\right]^{\frac{1}{2}}} = 0.006927 \text{ rad}$$

$$\text{and} \quad \phi = \tan^{-1}\left(\frac{62.5 (104.72)}{3125.0 - 1.4583 (104.72^2)}\right) = -0.4705 \text{ rad} = -26.9606^\circ$$

3.48 Find the steady-state response of the system shown in Fig. 3.55 for the following data: $k_1 = 1000$ N/m, $k_2 = 500$ N/m, $c = 500$ N-s/m, $m = 10$ kg, $r = 5$ cm, $J_0 = 1$ kg-m^2, $F_0 = 50$ N, $\omega = 20$ rad/s.





Equation of motion for rotation of pulley about O:

$$-k_2\,(\theta\, r)\, r - J_0\,\ddot{\theta} - k_1\, x\,(2\,r) - c\,\dot{x}\,(2\,r) + F_0 \sin\omega\, t\,(2\,r) - m\,\ddot{x}\,(2\,r) = 0 \qquad (1)$$

where $\theta = x/(2r)$. Equation (1) can be rearranged as:

$$\left(\frac{J_0}{2\,r} + 2\,m\,r\right)\ddot{x} + 2\,c\,r\,\dot{x} + \left(2\,k_1\,r + \frac{1}{2}\,k_2\,r\right)x = 2\,r\,F_0 \sin\omega\, t \qquad (2)$$

For given data, Eq. (2) becomes

$$11\,\ddot{x} + 50\,\dot{x} + 112.5\,x = 5 \sin 20\, t \qquad (3)$$

Steady state response is given by Eq. (3.25):

$$x_p(t) = X \cos(\omega\, t - \phi)$$

$$\text{where}\quad X = \frac{5}{\left[\left\{112.5 - 11\,(20^2)\right\}^2 + \left\{50\,(20)\right\}^2\right]^{\frac{1}{2}}} = 0.001136 \text{ m}$$

$$\text{and}\quad \phi = \tan^{-1}\left(\frac{50\,(20)}{112.5 - 11\,(20^2)}\right) = -0.2291 \text{ rad} = -13.1287^\circ$$

3.49 A uniform slender bar of mass m may be supported in one of two ways as shown in Fig. 3.56. Determine the arrangement that results in a reduced steady-state response of the bar under a harmonic force, $F_0 \sin \omega t$, applied at the middle of the bar, as shown in the figure.

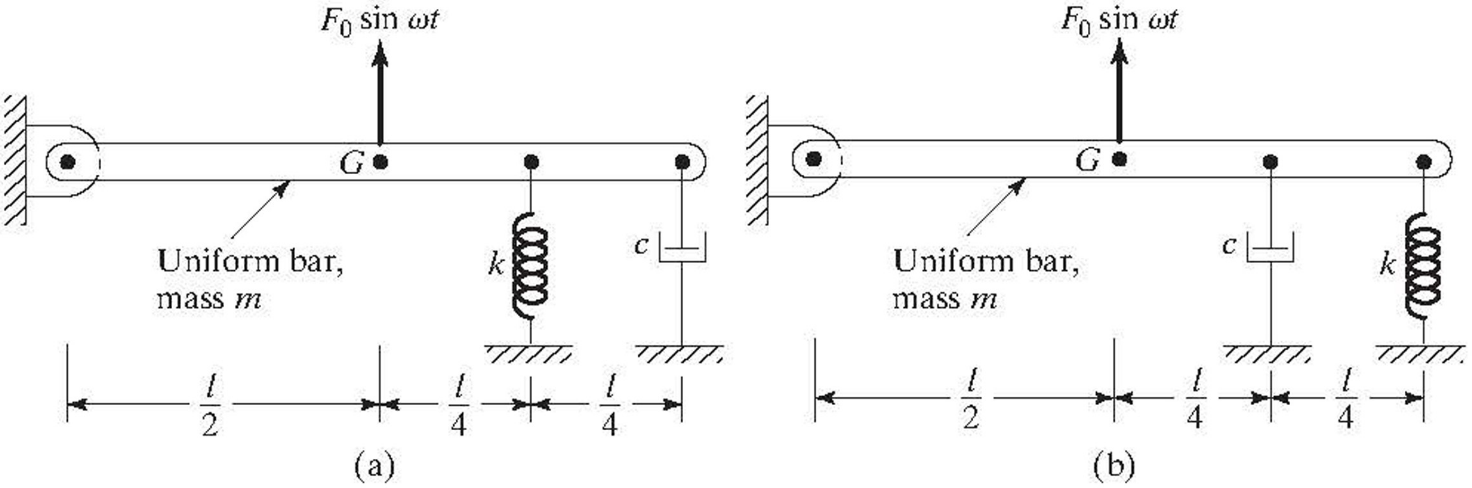


FIGURE 3.56

(a)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0 \ddot{\theta} + \left(k\,\theta\,\frac{3\,\ell}{4}\right)\frac{3\,\ell}{4} + (c\,\ell\,\dot{\theta})\,\ell = \frac{\ell}{2}\,F_0 \sin \omega t$$

$$\text{or} \quad I_0 \ddot{\theta} + c\,\ell^2\,\dot{\theta} + \frac{9}{16}\,k\,\ell^2\,\theta = \frac{F_0\,\ell}{2}\sin \omega t$$

Magnitude of steady state response:

$$\Theta_a = \left(\frac{F_0\,\ell}{2}\right) \Big/ \left[\left\{\frac{9}{16}\,k\,\ell^2 - I_0\,\omega^2\right\}^2 + (c\,\ell^2\,\omega)^2\right]^{\frac{1}{2}} \qquad (1)$$

(b)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0 \ddot{\theta} + (k\,\ell\,\theta)\,\ell + \left(c\,\frac{3\,\ell}{4}\,\dot{\theta}\right)\frac{3\,\ell}{4} = \frac{\ell}{2}\,F_0 \sin \omega t$$

$$\text{or} \quad I_0 \ddot{\theta} + \frac{9}{16}\,c\,\ell^2\,\dot{\theta} + k\,\ell^2\,\theta = \frac{F_0\,\ell}{2}\sin \omega t$$

Magnitude of steady state response:

$$\Theta_b = \left(\frac{F_0\,\ell}{2}\right) \Big/ \left[\left\{k\,\ell^2 - I_0\,\omega^2\right\}^2 + \left\{\frac{9}{16}\,c\,\ell^2\,\omega\right\}^2\right]^{\frac{1}{2}} \qquad (2)$$

Usually, c is small compared to k. If the term containing c is negligible, Θ_a will be smaller than Θ_b. Hence arrangement (a) is desirable.

Section 3.6 Response of a System Under the Harmonic Motion of the Base

3.52 A single-story building frame is subjected to a harmonic ground acceleration, as shown in Fig. 3.57. Find the steady-state motion of the floor (mass m).

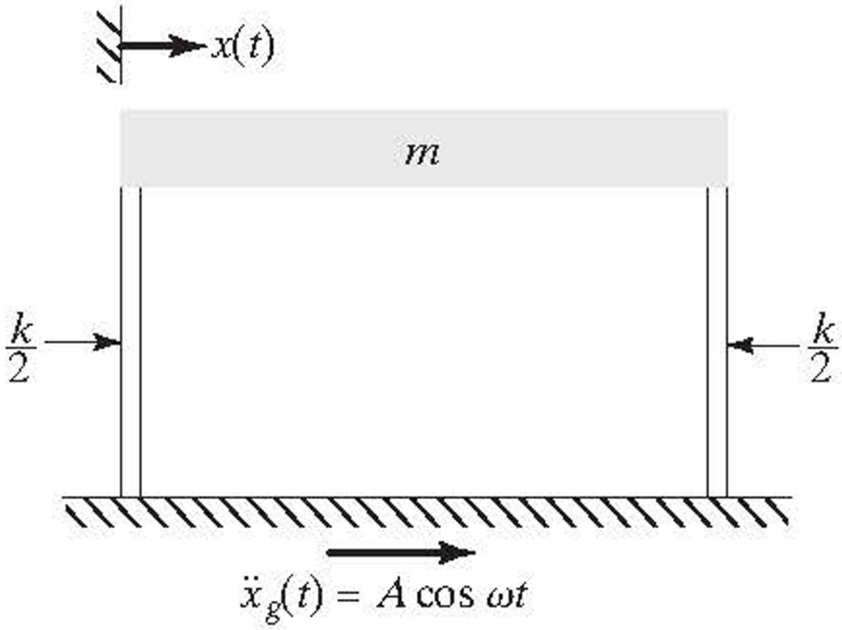
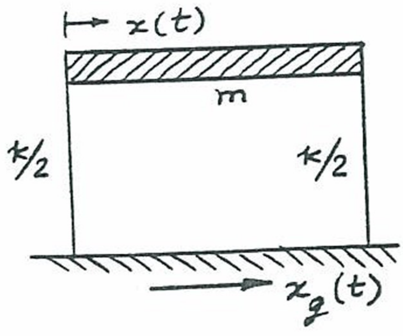


FIGURE 3.57

$\ddot{y}(t) = \ddot{x}_g(t) = A \cos \omega t \quad ; \quad \dot{y}(t) = \frac{A}{\omega} \sin \omega t + B_1$

$y(t) = -\frac{A}{\omega^2} \cos \omega t + B_1 t + B_2$



Assuming $y(0) = \dot{y}(0) = 0$, we get

$y(t) = -\frac{A}{\omega^2} \cos \omega t$

Equation of motion:

$m \ddot{x} + k(x - y) = 0$

i.e., $m \ddot{z} + k z = -m \ddot{y} = -m \ddot{x}_g(t) = -mA \cos \omega t$

where $z = x - y$

Solution is :

$$z(t) = \frac{-mA \cos \omega t}{k - m \omega^2}$$

$$\therefore \quad x(t) = z(t) + y(t) = -\left(\frac{m}{k - m\omega^2} + \frac{1}{\omega^2} \right) A \cos \omega t$$

3.53 Find the horizontal displacement of the floor (mass m) of the building frame shown in Fig. 3.57 when the ground acceleration is given by $\ddot{x}_g = 100 \sin \omega t$ mm/sec: Assume $m = 2000$ kg, $k = 0.1$ MN/m, $\omega = 25$ rad/s, and $x_g(t=0) = \dot{x}_g(t=0) = x(t=0) = \dot{x}(t=0) = 0$.

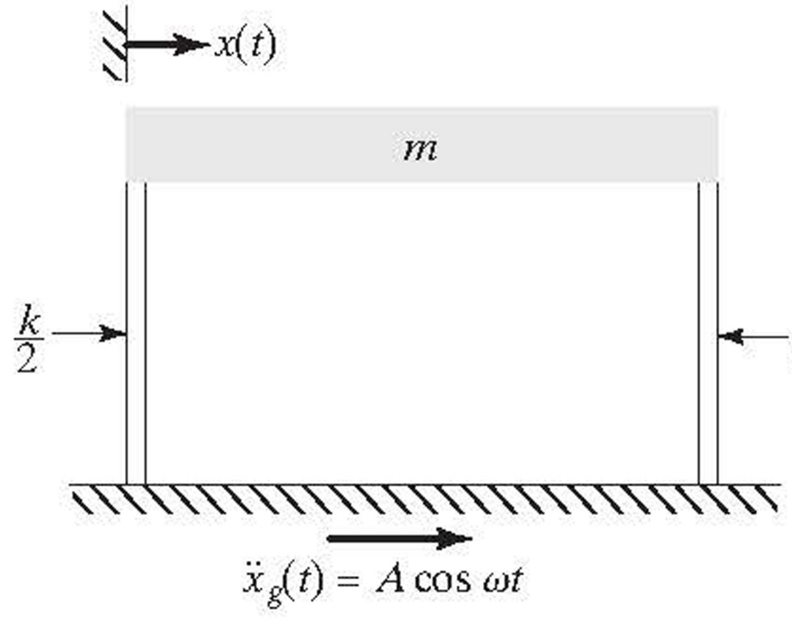


FIGURE 3.57

From solution of problem 3.52,

$$x(t) = \left| \frac{-mA}{k - m\omega^2} \right| \sin \omega t - \frac{A}{\omega^2} \sin \omega t$$

$$= \left| \frac{-2000\left(\frac{100}{1000}\right)}{0.1 * 10^6 - 2000\,(25)^2} \right| \sin 25t - \left(\frac{100}{1000}\right)\frac{1}{(25)^2} \sin 25t$$

For maximum $x(t)$,

$$x(t) = \left(\frac{-200}{1.15 \times 10^6} - \frac{1}{6250} \right) \sin 25t = -3.3391 \times 10^{-4} \sin 25t \ \text{m}$$

$\therefore$ Maximum horizontal displacement of floor = 0.3339 mm

3.54 If the ground in Fig. 3.57, is subjected to a horizontal harmonic displacement with frequency $\omega = 200$ rad/s and amplitude $X_g = 15$ mm, find the amplitude of vibration of the floor (mass m). Assume the mass of the floor as 2000 kg and the stiffness of the columns as 0.5 MN/m.

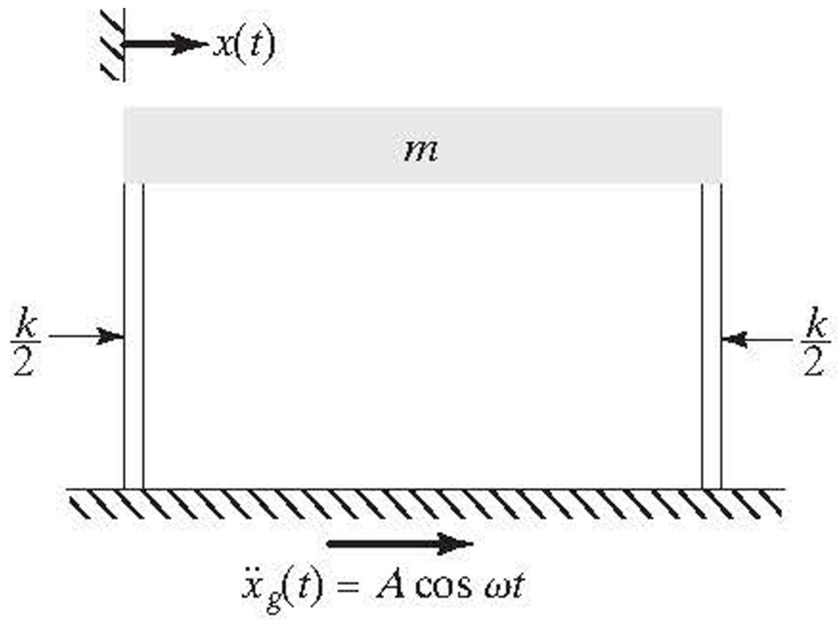


FIGURE 3.57

$$m(\ddot{x}-\ddot{y}) + k(x-y) = -m\ddot{y} = -m\ddot{x}_g \qquad (E_1)$$

Here $\quad y(t) = x_g(t) = X_g \cos\omega t$, and Eq. (E_1) becomes

$$m\ddot{z} + kz = m\omega^2 X_g \cos\omega t \qquad \text{with } z = x - y$$

Solution is:

$$z(t) = \frac{m\omega^2 X_g \cos\omega t}{k - m\omega^2} = \frac{X_g r^2 \cos\omega t}{1-r^2}$$

with $\quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.5\times10^6}{2000}} = 15.8114$ rad/sec

and $\quad r = \omega/\omega_n = 200/15.8114 = 12.6491$

$$z(t) = \left(\frac{15}{1000}\right)\left\{\frac{12.6491^2}{1-12.6491^2}\right\}\cos 200t = -0.01509 \cos 200t \text{ m}$$

$$x(t) = y(t) + z(t) = \{0.015 \cos 200t - |0.01509| \cos 200t\} \text{ m}$$

$\therefore$ Amplitude of vibration of floor $= 0.03009$ m $= 30.09$ mm.

3.58 A uniform bar of mass m is pivoted at point O and supported at the ends by two springs, as shown in Fig. 3.59. End P of spring PQ is subjected to a sinusoidal displacement, $x(t) = x_0 \sin \omega t$. Find the steady-state angular displacement of the bar when $l = 1$ m, $k = 1000$ N/m, $m = 10$ kg, $x_0 = 1$ cm, and $\omega = 10$ rad/s.

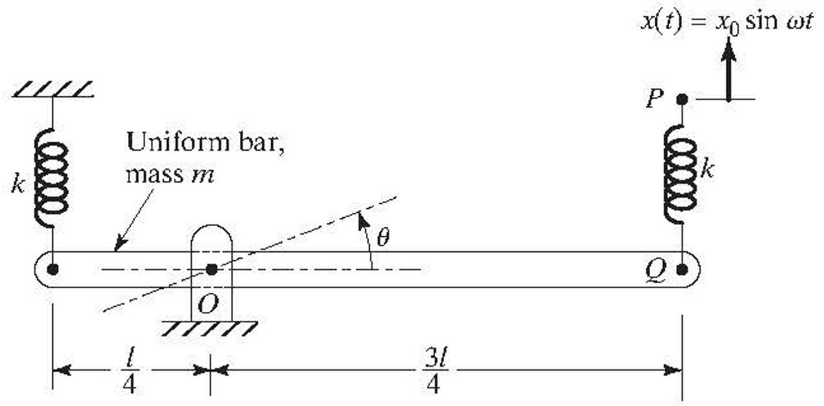
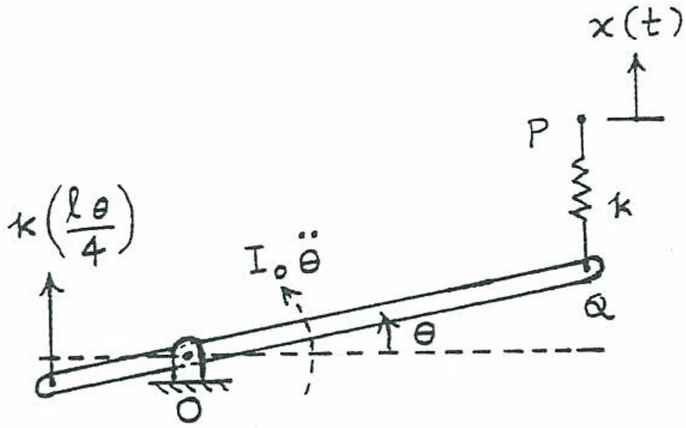


FIGURE 3.59



Linear displacement of point Q due to $\theta = \frac{3\,\ell}{4}\,\theta$ and net compression of spring PQ $= \frac{3}{4}\,\ell\,\theta - x(t)$. Equation of motion:

$$I_0\,\ddot{\theta} = -\,\frac{k\,\ell\,\theta}{4}\,\frac{\ell}{4} - k\left(\frac{3\,\ell\,\theta}{4} - x(t)\right)\frac{3\,\ell}{4} \tag{1}$$

where $I_0 = \frac{1}{12}\,m\,\ell^2 + m\,(\frac{\ell}{4})^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583$ kg–m^2

Hence Eq. (1) can be rewritten as

$$I_0\,\ddot{\theta} + \left(\frac{5}{8}\,k\,\ell^2\right)\theta = \left(\frac{3}{4}\,k\,\ell\,x_0\right)\sin\omega\,t \tag{2}$$

Steady state angular displacement of the bar is given by Eq. (3.6):

$$\Theta = \left(\frac{3}{4}\,k\,\ell\,x_0\right)\Big/\left(\frac{5}{8}\,k\,\ell^2 - I_0\,\omega^2\right) \tag{3}$$

$$= \left(\frac{3}{4}\,(1000)\,(1)\,(0.01)\right)\Big/\left(\frac{5}{8}\,(1000)\,(1^2) - 1.4583\,(10^2)\right) = 0.01565 \text{ rad}$$

and hence $\theta(t) = \Theta\,\sin\omega\,t = 0.01565\,\sin 10\,t$ rad

3.59 A uniform bar of mass m is pivoted at point O and supported at the ends by two springs, as shown in Fig. 3.60. End P of spring PQ is subjected to a sinusoidal displacement, $x(t) = x_0 \sin \omega t$. Find the steady-state angular displacement of the bar when $l = 1$ m, $k = 1000$ N/m, $c = 500$ N-s/m, $m = 10$ kg, $x_0 = 1$ cm, and $\omega = 10$ rad/s.

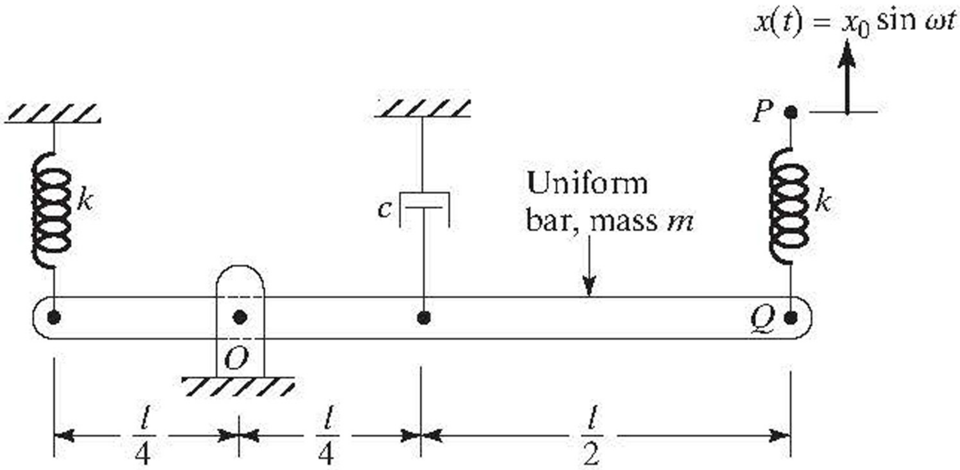


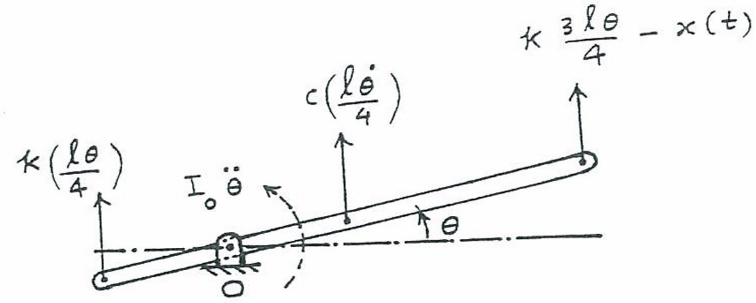
FIGURE 3.60

Equation of motion:

$$I_0 \ddot{\theta} = -k \frac{\ell \theta}{4} \left(\frac{\ell}{4}\right) - c \frac{\ell}{4} \dot{\theta} \left(\frac{\ell}{4}\right) - k \left[\frac{3\ell}{4} \theta - x(t)\right] \frac{3\ell}{4}$$

$$\text{i.e.,} \quad I_0 \ddot{\theta} + \frac{1}{16} c \ell^2 \dot{\theta} + \frac{5}{8} k \ell^2 = \frac{3}{4} k \ell x(t) = \frac{3}{4} k \ell x_0 \sin \omega t \tag{1}$$

$$\text{where} \quad I_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10)(1^2) = 1.4583 \text{ kg-m}^2 \tag{2}$$



Using given data, Eq. (1) can be expressed as

$$1.4583 \ddot{\theta} + \frac{1}{16} (500)(1^2) \dot{\theta} + \frac{5}{8} (1000)(1^2) \theta = \frac{3}{4} (1000)(1)(0.01) \sin 10 t$$

$$\text{i.e.,} \quad 1.4583 \ddot{\theta} + 31.25 \dot{\theta} + 625.0 \theta = 7.5 \sin 10 t \tag{3}$$

Steady state angular displacement of the bar is given by Eq. (3.28) with:

$$\Theta = \frac{7.5}{\left\{\left[625.0 - 1.4583 (10^2)\right]^2 + 31.25^2 (10^2)\right\}^{\frac{1}{2}}} = 0.01311 \text{ rad}$$

$$\phi = \tan^{-1} \left(\frac{31.25 (10)}{625.0 - 1.4583 (10^2)}\right) = 0.5779 \text{ rad}$$

$$\therefore \theta(t) = \Theta \sin (\omega t - \phi) = 0.01311 \sin (10 t - 0.5779) \text{ rad}$$

Section 3.7 Response of a Damped System Under Rotating Unbalance

3.63 A single-cylinder air compressor of mass 100 kg is mounted on rubber mounts, as shown in Fig. 3.61. The stiffness and damping constants of the rubber mounts are given by 10^6 N/m and 2000 N-s/m, respectively. If the unbalance of the compressor is equivalent to a mass 0.1 kg located at the end of the crank (point A), determine the response of the compressor at a crank speed of 3000 rpm. Assume $r = 10$ cm and $l = 40$ cm.

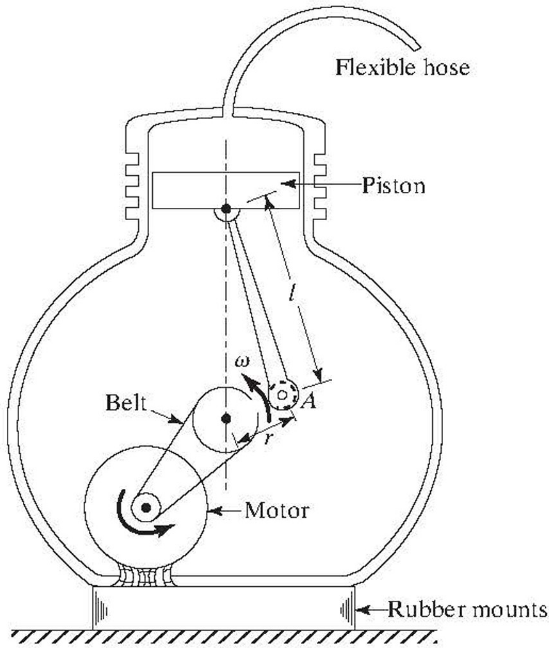


FIGURE 3.61

Equation of motion: $M \ddot{x} + c \dot{x} + k x = m e \omega^2 \sin \omega t$

where $\omega = \dfrac{3000 (2 \pi)}{60} = 314.16$ rad/sec, M = 100 kg, c = 2000 N-s/m, k = 10^6 N/m, m = 0.1 kg and e = r = 0.1 m. Steady state response is:

$$x_p(t) = X \sin (\omega t - \phi)$$

$$\text{where} \quad X = \frac{m e \omega^2}{\left[(k - M \omega^2)^2 + (c \omega)^2 \right]^{\frac{1}{2}}}$$

$$= \frac{0.1 (0.1) (314.16^2)}{\left[\left\{ 10^6 - 100 (314.16^2) \right\}^2 + (2000 (314.16))^2 \right]^{\frac{1}{2}}} = 110.9960 (10^{-6}) \text{ m}$$

$$\text{and} \quad \phi = \tan^{-1} \left(\frac{c \omega}{k - M \omega^2} \right) = \tan^{-1} \left(\frac{2000 (314.16)}{10^6 - 100 (314.16^2)} \right)$$

$$= - 0.07072 \text{ rad} = - 4.0520^\circ$$

3.64 One of the tail rotor blades of a helicopter has an unbalanced mass of $m = 0.5$ kg at a distance of $e = 0.15$ m from the axis of rotation, as shown in Fig. 3.62. The tail section has a length of 4 m, a mass of 240 kg, a flexural stiffness (EI) of 2.5 MN-m^2, and a damping ratio of 0.15. The mass of the tail rotor blades, including their drive system, is 20 kg. Determine the forced response of the tail section when the blades rotate at 1500 rpm.

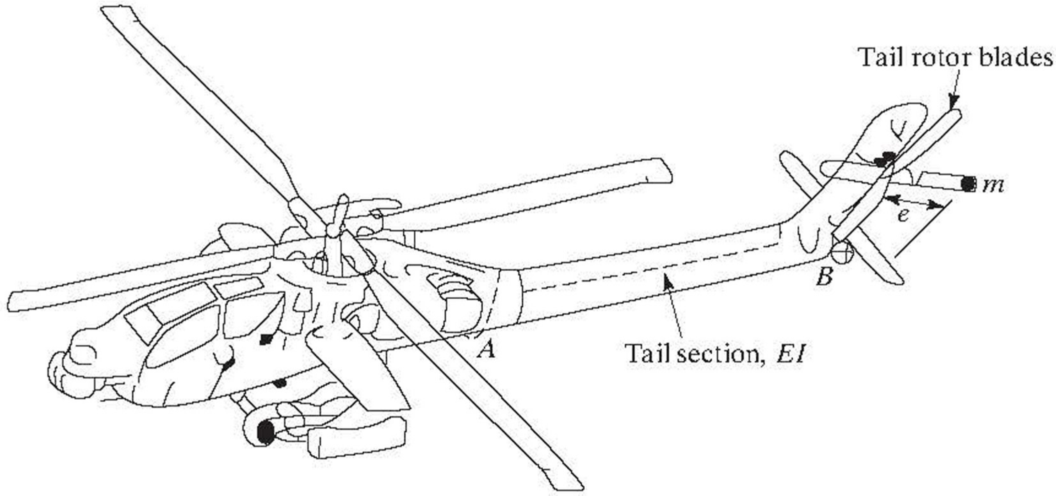
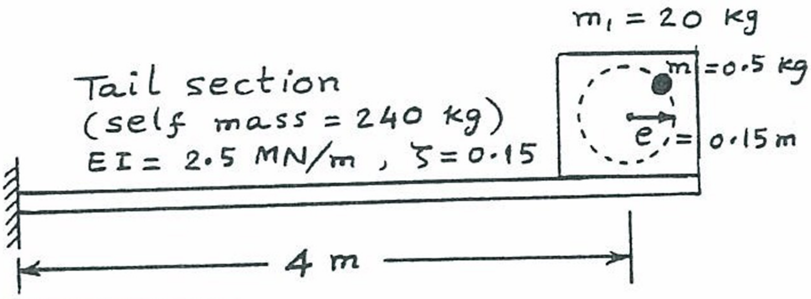


FIGURE 3.62

k = spring constant of cantilever beam

$$= \frac{3EI}{l^3} = \frac{3(2.5 \times 10^6)}{4^3}$$

$$= 0.1172 \times 10^6 \text{ N/m}$$



$$\omega_n = \sqrt{\frac{k}{m_1 + 0.25\, m_b}} = \sqrt{\frac{0.1172 \times 10^6}{20 + 0.25(240)}} = 38.2753 \text{ rad/sec}$$

$$\omega = 2\pi(1500)/60 = 157.08 \text{ rad/sec}$$

$$r = \omega/\omega_n = 157.08/38.2753 = 4.1040, \quad r^2 = 16.8428$$

Forced response is given by Eq. (3.79):

$$x_p(t) = X \sin(\omega t - \phi)$$

where

$$X = \frac{m\,e}{m_1} \cdot \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$= \frac{(0.5)(0.15)}{20} \; \frac{16.8428}{\sqrt{(1 - 16.8428)^2 + (2 \times 0.15 \times 4.1040)^2}}$$

$$= 3.9747 \times 10^{-3} \text{ m} = 3.9747 \text{ mm}$$

3.65 When an exhaust fan of mass 380 kg is supported on springs with negligible damping, the resulting static deflection is found to be 45 mm. If the fan has a rotating unbalance of 0.15 kg-m, find (a) the amplitude of vibration at 1750 rpm, and (b) the force transmitted to the ground at this speed.

$$\delta_{st} = \frac{45}{1000}\ m = \frac{Mg}{k} = \frac{380 \times 9.81}{k}$$

i.e., $k = 82,840$ N/m

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{82,840}{380}} = 14.7648 \text{ rad/sec} \ ; \quad \omega = \frac{2\pi(1750)}{60} = 183.26 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{183.26}{14.7648} = 12.412 \ ; \quad r^2 = 154.0566$$

(i) Amplitude of vibration

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.15}{380} \frac{154.0566}{\sqrt{(153.0566)^2 + 0}}$$

$$= 3.9732 \times 10^{-4} \text{ m}$$

(ii) Force transmitted to ground

$$= kX = (82840)(3.9732 \times 10^{-4}) = 32.9140 \text{ N}$$

3.66 A fixed-fixed steel beam, of length 5 m, width 0.5 m, and thickness 0.1 m, carries an electric motor of mass 75 kg and speed 1200 rpm at its mid-span, as shown in Fig. 3.63. A rotating force of magnitude $F_0 = 5000$ N is developed due to the unbalance in the rotor of the motor. Find the amplitude of steady-state vibrations by disregarding the mass of the beam. What will be the amplitude if the mass of the beam is considered?

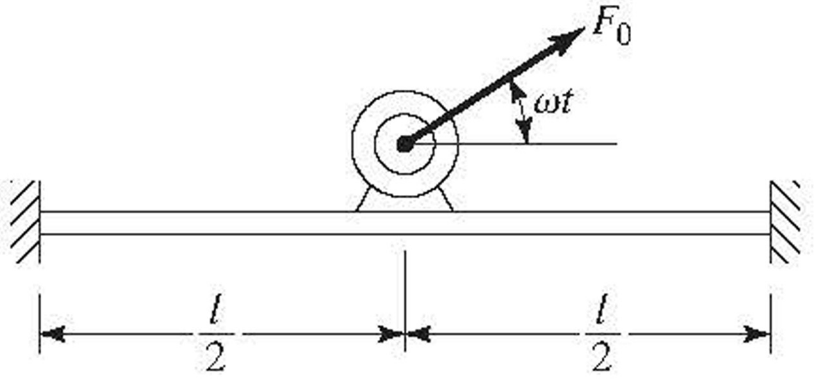


FIGURE 3.63

$$I = \frac{1}{12}(0.5)(0.1)^3 = 0.4167 \times 10^{-4}\ m^4$$

$$k = \frac{192\,EI}{l^3} = \frac{192\,(2.07\times10^{11})(0.4167\times10^{-4})}{(5)^3} = 1.3248\times10^7\ N/m$$

(a) $$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{13.248\times10^6}{75}} = 420.2856 \text{ rad/sec}$$

$$\omega = 2\pi(1200)/60 = 125.664 \text{ rad/sec}$$

$$r = \omega/\omega_n = 125.664/420.2856 = 0.299, \quad r^2 = 0.0894$$

Amplitude of steady-state vibration is given by Eq. (3.30) with $\zeta = 0$:

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k|(r^2-1)|} = \frac{5000}{(1.3248\times10^7)(0.9106)}$$
$$= 0.4145 \times 10^{-3}\ m$$

(b) Using the effective mass due to self weight of beam (for a cantilever) to be valid here also,

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\,m}}$$

where M = mass of motor = 75 kg, and

$$m = \text{mass of beam} = (5\times0.5\times0.1)\left(\frac{76.5\times10^3}{9.81}\right) = 1949.5313 \text{ kg}$$

$$\omega_n = \sqrt{\frac{13.248\times10^6}{75 + (1949.5313)(0.2357)}} = 157.4339 \text{ rad/sec}$$

$$r = \omega/\omega_n = 125.664/157.4339 = 0.7982, \quad r^2 = 0.6371$$

$$X = \frac{\delta_{st}}{|r^2-1|} = \frac{F_0}{k\,|r^2-1|} = \frac{5000}{(1.3248\times10^7)(0.3629)}$$
$$= 1.0400 \times 10^{-3}\ m$$

3.68 A centrifugal pump, weighing 600 N and operating at 1000 rpm, is mounted on six springs of stiffness 6000 N/m each. Find the maximum permissible unbalance in order to limit the steady-state deflection to 5 mm peak-to-peak.

$m = (600/9.81)$ N , $\omega = 2\pi(1000)/60 = 104.72$ rad/sec

$k = 6\,(6000) = 36{,}000$ N/m

$$\omega_n = \sqrt{k/m} = \sqrt{36\,000/\left(\frac{600}{9.81}\right)} = 24.2611 \text{ rad/sec}$$

$r = \omega/\omega_n = 104.72/24.2611 = 4.3164$, $r^2 = 18.6311$

$$X = \frac{F_0}{k\,|r^2-1|} = \frac{m_0\, e\, \omega^2}{k\,|r^2-1|}$$ where m_0 = unbalanced mass and e = eccentricity

i.e.,
$$2.5 \times 10^{-3} = \frac{m_0\, e\,(104.72)^2}{36000\,|17.6311|}$$

i.e., $m_0 e = 0.1447$ kg-m

$\therefore$ Unbalance $= W_0 e = m_0 g e = 0.1447\,(9.81) = 1.4195$ N-m

3.70 A variable-speed electric motor, having an unbalance, is mounted on an isolator. As the speed of the motor is increased from zero, the amplitudes of vibration of the motor are observed to be 0.55 in. at resonance and 0.15 in. beyond resonance. Find the damping ratio of the isolator.

Eq. (3.82): $$\frac{M X}{m e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

When $r = 1$,

$$\frac{M X}{m e} = \frac{1}{2\zeta} \quad \text{or} \quad \frac{M}{m e} = \frac{1}{2\zeta X} = \frac{1}{2\zeta (0.55)} = \frac{1}{1.1\zeta} \qquad (E_1)$$

When $r =$ large,

$$\frac{M X}{m e} \approx 1 \quad \text{or} \quad \frac{M}{m e} \approx \frac{1}{X} = \frac{1}{0.15} \qquad (E_2)$$

Combining (E_1) and (E_2), we obtain

$$\frac{M}{m e} = \frac{1}{0.15} = \frac{1}{1.1\zeta}$$

$$\therefore \quad \zeta = 0.1364$$

3.82 A spring-mass system is subjected to Coulomb damping. When a harmonic force of amplitude 120 N and frequency 2.5173268 Hz is applied, the system is found to oscillate with an amplitude of 75 mm. Determine the coefficient of dry friction if $m = 2$ kg and $k = 2100$ N/m.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2100}{2}} = 32.403703 \text{ rad/sec}$$

$$N = \text{vertical force} = mg = 2\,(9.81) = 19.62 \text{ N}$$

$$\frac{\omega}{\omega_n} = \frac{2.5173268 \times 2\pi}{32.403703} = 0.4881191$$

$$X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}\right]^{1/2}$$

i.e., $$0.075 = \frac{120}{2100}\left[\frac{1 - \left\{\frac{4\mu(19.62)}{\pi(120)}\right\}^2}{(1 - 0.4881191^2)}\right]^{1/2}$$

i.e., $$1.3125 = \left(\frac{1 - 0.04334\,\mu^2}{0.5802473}\right)^{1/2}$$

i.e., $$0.9995666 = 1 - 0.04334\,\mu^2$$

i.e., $$\mu = 0.1$$

Section 5.3 Free-Vibration Analysis of an Undamped System

5.5 Find the natural frequencies of the system shown in Fig. 5.24, with $m_1 = m$, $m_2 = 2m$, $k_1 = k$, and $k_2 = 2k$. Determine the response of the system when $k = 1000$ N/m, $m = 20$ kg, and the initial values of the displacements of the masses m_1 and m_2 are 1 and -1, respectively.

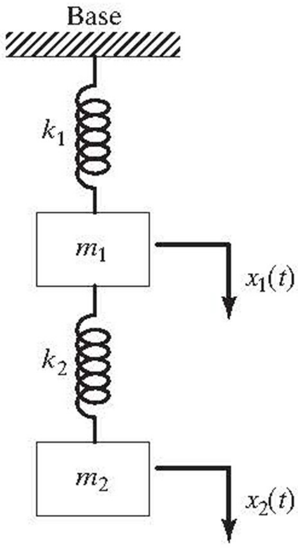


FIGURE 5.24

Equations of motion

$$\left.\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2)\, x_1 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 &= 0 \end{aligned}\right\} \qquad (E_1)$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, Eqs. (E_1) give the frequency equation

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

or

$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \qquad (E_2)$$

Roots of Eq. (E_2) are

$$\omega_1^2 ,\ \omega_2^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}} \qquad (E_3)$$

If $\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} = r_1 X_1^{(1)} \end{Bmatrix}$ and $\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} = r_2 X_1^{(2)} \end{Bmatrix}$,

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1 \omega_1^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_1^2 + k_2} \qquad (E_4)$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1 \omega_2^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_2^2 + k_2} \qquad (E_5)$$

General solution of (E_1) is

$$\left.\begin{aligned} x_1(t) &= X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ x_2(t) &= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{aligned}\right\} \qquad (E_6)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 can be found using Eqs. (5.18).

For $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$, (E_3) gives

$$\omega_1^2 = (2 - \sqrt{3}) \frac{k}{m} , \qquad \omega_2^2 = (2 + \sqrt{3}) \frac{k}{m} \qquad (E_7)$$

5.6 Set up the differential equations of motion for the double pendulum shown in Fig. 5.25, using the coordinates x_1 and x_2 and assuming small amplitudes. Find the natural frequencies, the ratios of amplitudes, and the locations of nodes for the two modes of vibration when $m_1 = m_2 = m$ and $l_1 = l_2 = l$.

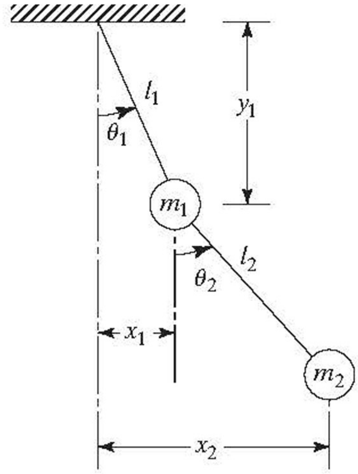
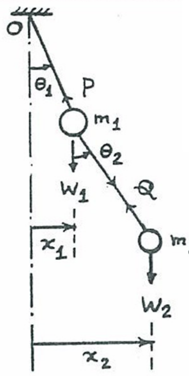


FIGURE 5.25

5.6 Taking moments about O and mass m_1,



$$m_1 l_1^2 \ddot{\theta}_1 = - W_1 (l_1 \sin\theta_1) + Q \sin\theta_2 (l_1 \cos\theta_1) - Q \cos\theta_2 (l_1 \sin\theta_1)$$

$$= - W_1 l_1 \theta_1 + W_2 l_1 (\theta_2 - \theta_1) \qquad (E_1)$$

assuming $Q \simeq W_2$.

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) = - W_2 (l_2 \sin\theta_2)$$

$$= - W_2 l_2 \theta_2 \qquad (E_2)$$

Using the relations $\theta_1 = \frac{x_1}{l_1}$ and $\theta_2 = \frac{x_2 - x_1}{l_2}$, Eqs. (E_1) and (E_2) become

$$m_1 l_1 \ddot{x}_1 + \left[W_1 + W_2\left(\frac{l_1 + l_2}{l_2}\right)\right] x_1 - \frac{W_2 l_1}{l_2} x_2 = 0 \qquad (E_3)$$

$$m_2 l_2 \ddot{x}_2 - W_2 x_1 + W_2 x_2 = 0 \qquad (E_4)$$

When $m_1 = m_2 = m$, $l_1 = l_2 = l$ and $W_1 = W_2 = mg$, Eqs. (E_3) and (E_4) give

$$m l \ddot{x}_1 + 3 mg\, x_1 - mg\, x_2 = 0$$

$$m l \ddot{x}_2 - mg\, x_1 + mg\, x_2 = 0 \qquad (E_5)$$

For harmonic motion $x_i(t) = X_i \cos \omega t$; $i = 1, 2$, Eqs. (E_5) become

$$- \omega^2 m l X_1 + 3 mg X_1 - mg X_2 = 0$$

$$- \omega^2 m l X_2 - mg X_1 + mg X_2 = 0 \qquad (E_6)$$

from which the frequency equation can be obtained as

$$\omega^4 m^2 l^2 - (4 m^2 l g)\, \omega^2 + 2 m^2 g^2 = 0$$

i.e. $\omega_1^2, \omega_2^2 = (2 \mp \sqrt{2}) \frac{g}{l}$

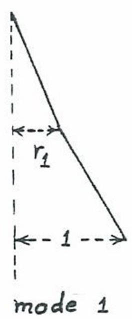
$\therefore \quad \omega_1 = 0.7654 \sqrt{\frac{g}{l}}, \quad \omega_2 = 1.8478 \sqrt{\frac{g}{l}}$

Ratio of amplitudes is given by Eq. (E_6) as

$$\frac{X_1}{X_2} = \frac{mg}{- \omega^2 m l + 3 mg} = \frac{1}{\left(- \omega^2 \frac{l}{g} + 3\right)}$$

In mode 1, $\omega_1 = 0.7654 \sqrt{\frac{g}{l}}$, $r_1 = \left(\frac{X_1}{X_2}\right)^{(1)} = 0.4142$

No node.

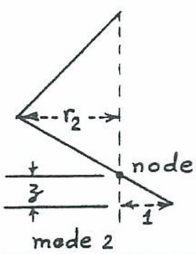


In mode 2, $\omega_2 = 1.8478 \sqrt{\frac{g}{l}}$,

$$r_2 = \left(\frac{X_1}{X_2}\right)^{(2)} = - 2.4133$$

one node located at z:

$$\frac{z}{1} = \frac{1 - z}{2.4133} \quad \text{or} \quad z = 0.2930$$



5.7 Determine the natural modes of the system shown in Fig. 5.26 when $k_1 = k_2 = k_3 = k$.

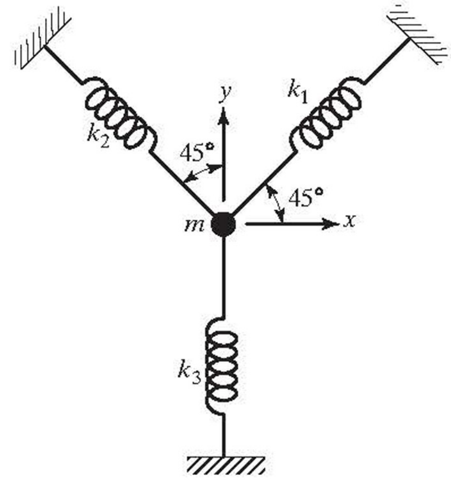
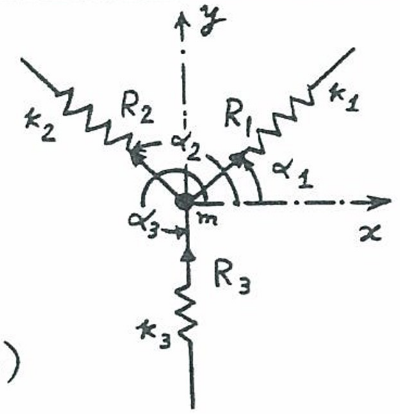


FIGURE 5.26

(5.7) Let R_1, R_2 and R_3 be the restoring forces in springs. Equations of motion of mass m in x and y directions are



$$m\ddot{x} = \sum_{i=1}^{3} R_i \cos\alpha_i \qquad (E_1)$$

$$m\ddot{y} = \sum_{i=1}^{3} R_i \sin\alpha_i \qquad (E_2)$$

where $R_i = -k_i\,(x\cos\alpha_i + y\sin\alpha_i)$ (E_3)

Eqs. (E_1) to (E_3) give

$$m\ddot{x} + \sum_{i=1}^{3} k_i\,(x\cos^2\alpha_i + y\sin\alpha_i\cos\alpha_i) = 0 \qquad (E_4)$$

$$m\ddot{y} + \sum_{i=1}^{3} k_i\,(x\sin\alpha_i\cos\alpha_i + y\sin^2\alpha_i) = 0 \qquad (E_5)$$

For $\alpha_1 = 45°$, $\alpha_2 = 135°$, $\alpha_3 = 270°$ and $k_1 = k_2 = k_3 = k$, Eqs. (E_4) and (E_5) reduce to

$$m\ddot{x} + kx = 0 \qquad (E_6)$$

$$m\ddot{y} + 2ky = 0 \qquad (E_7)$$

These equations are uncoupled. For harmonic motion, $x(t) = X\cos(\omega t + \phi)$, $y(t) = Y\cos(\omega t + \phi)$, and hence

$\omega_1 = \sqrt{\frac{k}{m}}$ for motion in x direction

$\omega_2 = \sqrt{\frac{2k}{m}}$ for motion in y direction

Natural modes are given by

$$x(t) = X\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right)$$

$$y(t) = Y\cos\left(\sqrt{\frac{2k}{m}}\,t + \phi_2\right)$$

where X, ϕ_1, Y and ϕ_2 can be determined from initial conditions.

5.8 A machine tool, having a mass of $m = 1000$ kg and a mass moment of inertia of $J_0 = 300$ kg-m^2, is supported on elastic supports, as shown in Fig. 5.27. If the stiffnesses of the supports are given by $k_1 = 3000$ N/mm and $k_2 = 2000$ N/mm, and the supports are located at $l_1 = 0.5$ m and $l_2 = 0.8$ m, find the natural frequencies and mode shapes of the machine tool.

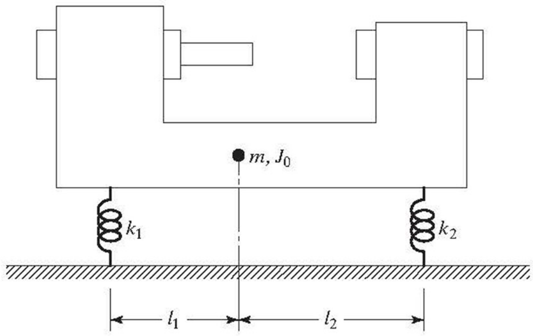
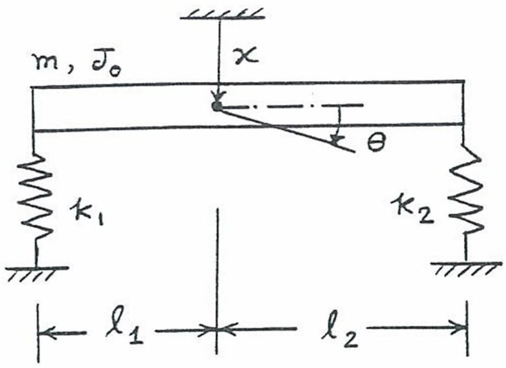


FIGURE 5.27

5.8 Equations of motion in terms of x and θ:



$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0 \qquad (E_1)$$

$$J_0\ddot{\theta} - k_1 l_1 (x - l_1\theta) + k_2 l_2 (x + l_2\theta) = 0 \qquad (E_2)$$

For free vibration,

$$x(t) = X \cos(\omega t + \phi) \qquad (E_3)$$

$$\theta(t) = \Theta \cos(\omega t + \phi) \qquad (E_4)$$

and Eqs. (E_1) and (E_2) become

$$\begin{bmatrix} -m\omega^2 + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_5)$$

Frequency equation is

$$\begin{vmatrix} -m\omega^2 + k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{vmatrix} = 0 \qquad (E_6)$$

i.e.,

$$\begin{vmatrix} -\omega^2 + 5000 & 100 \\ 100 & -0.3\,\omega^2 + 2030 \end{vmatrix} = 0$$

i.e.,

$$0.3\,\omega^4 - 3530\,\omega^2 + 10.14 \times 10^6 = 0$$

i.e.,

$$\omega^2 = 6785.3373, \quad 4981.3293$$

$$\therefore \quad \omega_1 = 70.5785 \text{ rad/sec}, \qquad \omega_2 = 82.3732 \text{ rad/sec}$$

Mode shapes:

$$(-1000\,\omega_1^2 + 5 \times 10^6)\, X + 0.1 \times 10^6\, \Theta = 0$$

or

$$\left.\frac{X}{\Theta}\right|_{\omega_1} = \frac{-0.1 \times 10^6}{-1000\,\omega_1^2 + 5 \times 10^6} = -5.3476$$

and

$$\left.\frac{X}{\Theta}\right|_{\omega_2} = \frac{-0.1 \times 10^6}{-1000\,\omega_2^2 + 5 \times 10^6} = 0.05601$$

5.31 Two identical pendulums, each with mass m and length l, are connected by a spring of stiffness k at a distance d from the fixed end, as shown in Fig. 5.35.

a. Derive the equations of motion of the two masses.

b. Find the natural frequencies and mode shapes of the system.

c. Find the free-vibration response of the system for the initial conditions $\theta_1(0) = a$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$, and $\dot{\theta}_2(0) = 0$.

d. Determine the condition(s) under which the system exhibits a beating phenomenon.

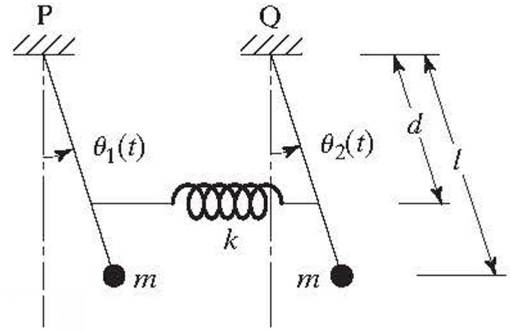
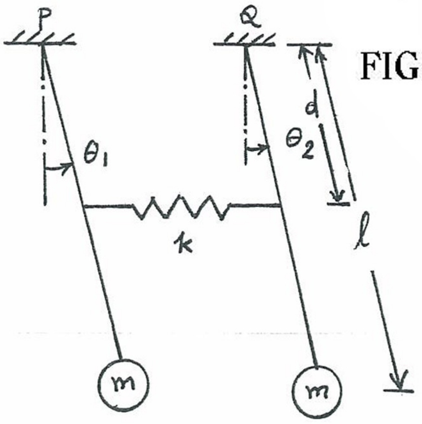


FIGURE 5.35

(a) Equations of motion:

Assume: θ_1, θ_2 are small.

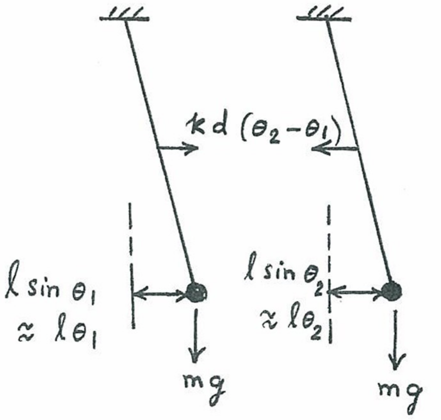
Moment equilibrium equations of the two masses about P and Q:



$$m l^2 \ddot{\theta}_1 + m g l \theta_1 + k d^2 (\theta_1 - \theta_2) = 0 \qquad (1)$$

$$m l^2 \ddot{\theta}_2 + m g l \theta_2 - k d^2 (\theta_1 - \theta_2) = 0 \qquad (2)$$

(b) Natural frequencies and mode shapes:



Free body diagram

Assume: Harmonic motion with

$$\theta_i(t) = \Theta_i \cos(\omega t - \phi); \; i = 1, 2 \qquad (3)$$

where Θ_1 and Θ_2 are amplitudes of θ_1 and θ_2, respectively, ω is the natural frequency, and ϕ is the phase angle.

Using Eq. (3), Eqs. (1) and (2) can be expressed in matrix form as

$$-\omega^2 m l^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} + \begin{bmatrix} mgl + kd^2 & -kd^2 \\ -kd^2 & mgl + kd^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4)$$

Frequency equation:

$$\begin{vmatrix} -\omega^2 m l^2 + mgl + kd^2 & -kd^2 \\ -kd^2 & -\omega^2 m l^2 + mgl + kd^2 \end{vmatrix} = 0$$

or

$$\omega^4 - \omega^2 \left(\frac{2g}{l} + \frac{2kd^2}{ml^2} \right) + \left(\frac{g^2}{l^2} + \frac{2gkd^2}{ml^3} \right) = 0 \qquad (5)$$

Solution of Eq. (5) gives

$$\omega_1^2 = \frac{g}{l}, \quad \omega_2^2 = \frac{g}{l} + \frac{2kd^2}{ml^2} \qquad (6)$$

By substituting for ω_1^2 and ω_2^2 into Eq. (4), we obtain

$$\left(\frac{\Theta_2}{\Theta_1}\right)^{(1)} = 1 \qquad \text{or} \qquad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)}$$

and

$$\left(\frac{\Theta_2}{\Theta_1}\right)^{(2)} = -1 \qquad \text{or} \qquad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \Theta_1^{(2)}$$

Thus the motion of the masses in the two modes is given by

$$\vec{\theta}^{(1)}(t) = \begin{Bmatrix} \theta_1^{(1)}(t) \\ \theta_2^{(1)}(t) \end{Bmatrix} = \Theta_1^{(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\omega_1 t + \phi_1) \qquad (7)$$

$$\vec{\theta}^{(2)}(t) = \begin{Bmatrix} \theta_1^{(2)}(t) \\ \theta_2^{(2)}(t) \end{Bmatrix} = \Theta_1^{(2)} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos(\omega_2 t + \phi_2) \qquad (8)$$

(c) Free vibration response:

Using linear superposition of natural modes, the free vibration response of the system is given by

$$\vec{\theta}(t) = c_1 \, \vec{\theta}^{(1)}(t) + c_2 \, \vec{\theta}^{(2)}(t) \qquad (9)$$

By choosing $c_1 = c_2 = 1$, with no loss of generality, Eqs. (7) to (9) lead to

$$\theta_1(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) + \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (10)$$

$$\theta_2(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) - \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (11)$$

where $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 are constants to be determined from the initial conditions. When $\theta_1(0) = a$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$ and $\dot{\theta}_2(0) = 0$, Eqs. (10) and (11) yield

$$\left.\begin{aligned} a &= \Theta_1^{(1)} \cos\phi_1 + \Theta_1^{(2)} \cos\phi_2 \\ 0 &= \Theta_1^{(1)} \cos\phi_1 - \Theta_1^{(2)} \cos\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 - \omega_2 \Theta_1^{(2)} \sin\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 + \omega_2 \Theta_1^{(2)} \sin\phi_2 \end{aligned}\right\} \qquad (12)$$

Eqs. (12) can be solved for $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 to obtain

$$\left.\begin{aligned} \theta_1(t) &= a \cos\frac{\omega_2 - \omega_1}{2} t \cdot \cos\frac{\omega_2 + \omega_1}{2} t \\ \theta_2(t) &= a \sin\frac{\omega_2 - \omega_1}{2} t \cdot \sin\frac{\omega_2 + \omega_1}{2} t \end{aligned}\right\} \qquad (13)$$

(d) Conditions for beating:

When

$$\frac{2 k d^2}{m l^2} \ll \frac{g}{l} \qquad \text{or} \qquad k \ll \frac{m g l}{2 d^2}, \qquad (14)$$

the two frequency components in Eqs. (13), namely, $\frac{\omega_2 - \omega_1}{2}$ and $\frac{\omega_2 + \omega_1}{2}$, can be approximated as

$$\Omega_1 = \frac{\omega_2 - \omega_1}{2} \simeq \frac{k}{2m} \frac{d^2}{\sqrt{g l^3}} \qquad (15)$$

and

$$\Omega_2 = \frac{\omega_2 + \omega_1}{2} \simeq \sqrt{\frac{g}{l}} + \frac{k}{2m} \frac{d^2}{\sqrt{g l^3}} \qquad (16)$$

This implies that the motions of the pendulums are given by

$$\left.\begin{aligned} \theta_1(t) &\simeq a \cos\Omega_1 t \cdot \cos\Omega_2 t \\ \theta_2(t) &\simeq a \sin\Omega_1 t \cdot \sin\Omega_2 t \end{aligned}\right\} \qquad (17)$$

This motion, Eqs. (17), denotes beating phenomenon.

5.36 Determine the natural frequencies and normal modes of the torsional system shown in Fig. 5.38 for $k_{t2} = 2k_{t1}$ and $J_2 = 2J_1$.

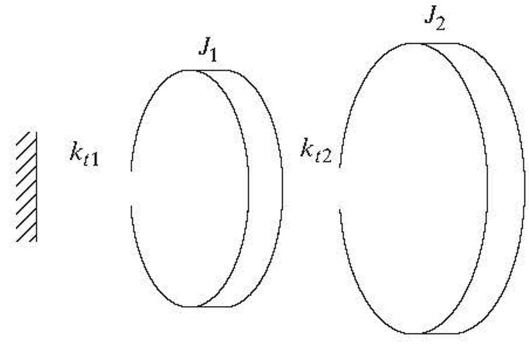


FIGURE 5.38

5.36 With $k_{t1} = k_t$, $k_{t2} = 2\,k_t$, $J_1 = J_0$, $J_2 = 2\,J_0$, $k_{t3} = 0$ and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 3\,k_t\,\theta_1 - 2\,k_t\,\theta_2 = 0$$

$$2\,J_0\,\ddot{\theta}_2 - 2\,k_t\,\theta_1 + 2\,k_t\,\theta_2 = 0$$

For harmonic solution, $\theta_i(t) = \Theta_i \cos(\omega t + \phi)\ ;\ i = 1, 2\,,$

$$\begin{bmatrix} (-\omega^2 J_0 + 3\,k_t) & -2\,k_t \\ -2\,k_t & (-2\omega^2 J_0 + 2\,k_t) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

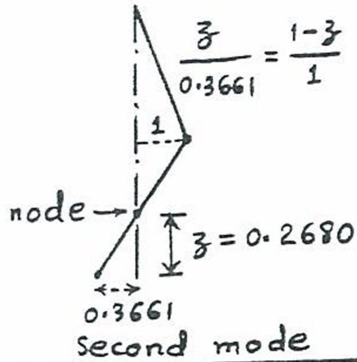
$$\begin{vmatrix} -\omega^2 J_0 + 3\,k_t & -2\,k_t \\ -2\,k_t & -2\,\omega^2 J_0 + 2\,k_t \end{vmatrix} = 2\,J_0^2\,\omega^4 - 8\,J_0\,k_t\,\omega^2 + 2\,k_t^2 = 0$$

$$\omega^2 = (2 \mp \sqrt{3})\,\frac{k_t}{J_0}\ ; \qquad \omega_1 = 0.5176\,\sqrt{\frac{k_t}{J_0}}\,, \qquad \omega_2 = 1.9319\,\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-J_0\,\omega_1^2 + 3\,k_t}{2\,k_t} = 1.3661$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-J_0\,\omega_2^2 + 3\,k_t}{2\,k_t} = -0.3661$$

First mode



Second mode

5.37 Determine the natural frequencies of the system shown in Fig. 5.39 by assuming that the rope passing over the cylinder does not slip.

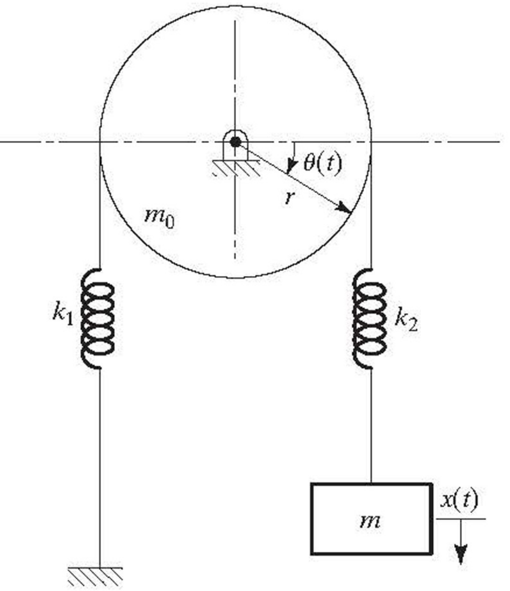


FIGURE 5.39

5.37 Equation of motion of mass m: $\quad m\ddot{x} = -k_2(x - r\theta) \quad \text{---}(E_1)$

Equation of motion of cylinder of mass m_0 and mass moment of inertia $J_0 = \frac{1}{2} m_0 r^2$: $\quad J_0 \ddot{\theta} = -k_1 r^2 \theta - k_2 (r\theta - x) r \quad \text{---}(E_2)$

For $x(t) = X \cos(\omega t + \phi)$ and $\theta(t) = \Theta \cos(\omega t + \phi)$, Eqs. (E_1) and (E_2) give the frequency equation

$$\begin{vmatrix} -m\omega^2 + k_2 & -k_2 r \\ -k_2 r & -\frac{1}{2} m_0 r^2 \omega^2 + k_1 r^2 + k_2 r^2 \end{vmatrix} = 0$$

i.e.

$$\omega^4 - \omega^2 \left(\frac{k_2}{m} + \frac{2\{k_1 + k_2\}}{m_0} \right) + \frac{2 k_1 k_2}{m_0 m} = 0$$

$$\omega_1^2, \omega_2^2 = \frac{k_2}{2m} + \frac{(k_1 + k_2)}{m_0} \mp \sqrt{\frac{1}{4}\left(\frac{k_2}{m} + \frac{2k_1}{m_0} + \frac{2k_2}{m_0} \right)^2 - \frac{2 k_1 k_2}{m\, m_0}}$$

5.38 Find the natural frequencies and mode shapes of the system shown in Fig. 5.8(a) by assuming that $J_1 = J_0$, $J_2 = 2J_0$, and $k_{t1} = k_{t2} = k_{t3} = k_t$.

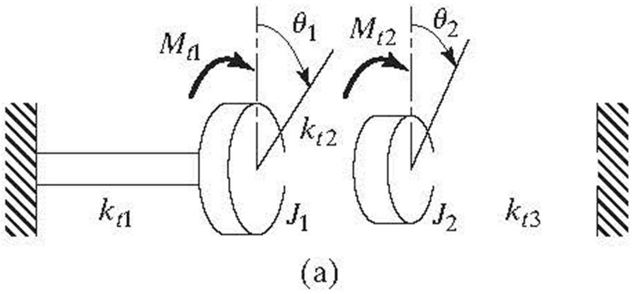


FIGURE 5.8 Torsional system with discs mounted on a shaft.

5.38 For $J_1 = J_0$, $J_2 = 2J_0$, $k_{t_1} = k_{t_2} = k_{t_3} = k_t$, and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0\,\ddot{\theta}_1 + 2k_t\,\theta_1 - k_t\,\theta_2 = 0$$

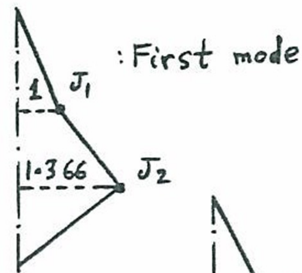
$$2J_0\,\ddot{\theta}_2 - k_t\,\theta_1 + 2k_t\,\theta_2 = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -\omega^2 J_0 + 2k_t & -k_t \\ -k_t & -2\omega^2 J_0 + 2k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

from which the frequency equation can be obtained as

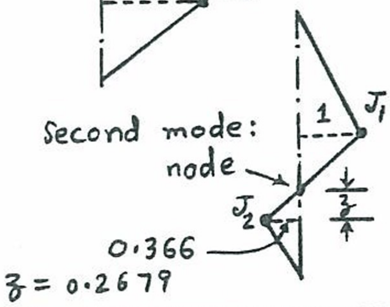
$$2J_0^2\omega^4 - 6J_0 k_t\,\omega^2 + 3k_t^2 = 0$$



$$\omega_1^2,\ \omega_2^2 = \frac{1}{2}(3 \mp \sqrt{3})\,\frac{k_t}{J_0}$$

$$\therefore\ \omega_1 = 0.79623\sqrt{\frac{k_t}{J_0}}\ ;\qquad \omega_2 = 1.53819\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 2k_t}{k_t} = 1.36603$$



$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 2k_t}{k_t} = -0.36603$$

5.39 Determine the normal modes of the torsional system shown in Fig. 5.9 when $k_{t1} = k_t$, $k_{t2} = 5k_t$, $J_1 = J_0$, and $J_2 = 5J_0$.

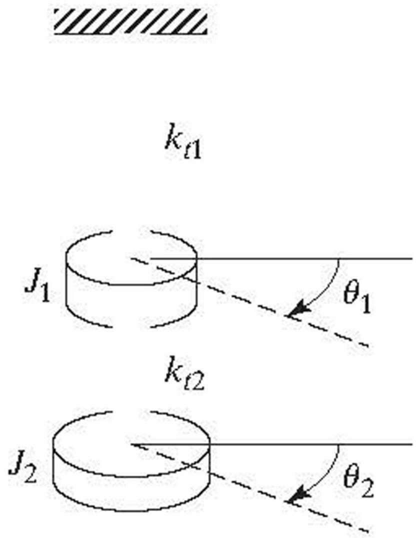


FIGURE 5.9
Torsional system.

(5.39) Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 6 k_t \theta_1 - 5 k_t \theta_2 = 0$$

$$5 J_0 \ddot{\theta}_2 - 5 k_t \theta_1 + 5 k_t \theta_2 = 0$$

These equations can be expressed as, for harmonic motion,

$$\begin{bmatrix} -\omega^2 J_0 + 6 k_t & -5 k_t \\ -5 k_t & -5 \omega^2 J_0 + 5 k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$J_0^2 \omega^4 - 7 k_t J_0 \omega^2 + k_t^2 = 0$$

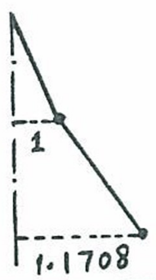
$$\omega_1^2, \omega_2^2 = \frac{k_t}{J_0} \left(\frac{7}{2} \mp \frac{1}{2} \sqrt{45} \right) = 0.1459 \frac{k_t}{J_0}, \quad 6.8541 \frac{k_t}{J_0}.$$

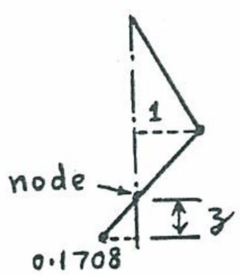
$$\omega_1 = 0.38197 \sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 2.61803 \sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 6 k_t}{5 k_t} = 1.1708$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 6 k_t}{5 k_t} = -0.1708$$

First mode





Second mode

$$\frac{z}{0.1708} = \frac{1-z}{1}$$

$$z = 0.1459$$