CHAPTER 6

EXPERIMENTAL MAPPING METHODS

We have seen in the last few chapters that the potential is the gateway to any information we desire about the electrostatic field at a point. The path is straight, and travel on it is easy in whichever direction we wish to go. The electric field intensity may be found from the potential by the gradient operation, which is a differentiation, and the electric field intensity may then be used to find the electric flux density by multiplying by the permittivity. The divergence of the flux density, again a differentiation, gives the volume charge density; and the surface charge density on any conductors in the field is quickly found by evaluating the flux density at the surface. Our boundary conditions show that it must be normal to such a surface.

Integration is still required if we need more information than the value of a field or charge density *at a point*. Finding the total charge on a conductor, the total energy stored in an electrostatic field, or a capacitance or resistance value are examples of such problems, each requiring an integration. These integrations cannot generally be avoided, no matter how extensive our knowledge of field theory becomes, and indeed, we should find that the greater this knowledge becomes, the more integrals we should wish to evaluate. Potential can do one important thing for us, and that is to quickly and easily furnish us with the quantity we must integrate.

Our goal, then, is to find the potential first. This cannot be done in terms of a charge configuration in a practical problem, because no one is kind enough to

tell us exactly how the charges are distributed. Instead, we are usually given several conducting objects or conducting boundaries and the potential difference between them. Unless we happen to recognize the boundary surfaces as belonging to a simple problem we have already disposed of, we can do little now and must wait until Laplace's equation is discussed in the following chapter.

Although we thus postpone the mathematical solution to this important type of practical problem, we may acquaint ourselves with several experimental methods of finding the potential field. Some of these methods involve special equipment such as an electrolytic trough, a fluid-flow device, resistance paper and the associated bridge equipment, or rubber sheets; others use only pencil, paper, and a good supply of erasers. The *exact* potential can never be determined, but sufficient accuracy for engineering purposes can usually be attained. One other method, called the *iteration* method, does allow us to achieve any desired accuracy for the potential, but the number of calculations required increases very rapidly as the desired accuracy increases.

Several of the experimental methods to be described below are based on an analogy with the electrostatic field, rather than directly on measurements on this field itself.

Finally, we cannot introduce this subject of experimental methods of finding potential fields without emphasizing the fact that many practical problems possess such a complicated geometry that no exact method of finding that field is possible or feasible and experimental techniques are the only ones which can be used.

6.1 CURVILINEAR SQUARES

Our first mapping method is a graphical one, requiring only pencil and paper. Besides being economical, it is also capable of yielding good accuracy if used skillfully and patiently. Fair accuracy (5 to 10 percent on a capacitance determination) may be obtained by a beginner who does no more than follow the few rules and hints of the art.

The method to be described is applicable only to fields in which no variation exists in the direction normal to the plane of the sketch. The procedure is based on several facts we have already demonstrated:

1. A conductor boundary is an equipotential surface.
2. The electric field intensity and electric flux density are both perpendicular to the equipotential surfaces.
3. **E** and **D** are therefore perpendicular to the conductor boundaries and possess zero tangential values.
4. The lines of electric flux, or streamlines, begin and terminate on charge and hence, in a charge-free, homogeneous dielectric, begin and terminate only on the conductor boundaries.

Let us consider the implications of these statements by drawing the streamlines on a sketch which already shows the equipotential surfaces. In Fig. 6.1*a* two conductor boundaries are shown, and equipotentials are drawn with a constant potential difference between lines. We should remember that these lines are only the cross sections of the equipotential surfaces, which are cylinders (although not circular), since no variation in the direction normal to the surface of the paper is permitted. We arbitrarily choose to begin a streamline, or flux line, at A on the surface of the more positive conductor. It leaves the surface normally and must cross at right angles the undrawn but very real equipotential surfaces between the conductor and the first surface shown. The line is continued to the other conductor, obeying the single rule that the intersection with each equipotential must be square. Turning the paper from side to side as the line progresses enables us to maintain perpendicularity more accurately. The line has been completed in Fig. 6.1*b*.

In a similar manner, we may start at B and sketch another streamline ending at B'. Before continuing, let us interpret the meaning of this pair of streamlines. The streamline, by definition, is everywhere tangent to the electric field intensity or to the electric flux density. Since the streamline is tangent to the electric flux density, the flux density is tangent to the streamline and no electric flux may cross any streamline. In other words, if there is a charge of 5 μC on the surface between A and B (and extending 1 m into the paper), then 5 μC of flux begins in this region and all must terminate between A' and B'. Such a pair of lines is sometimes called a flux *tube*, because it physically seems to carry flux from one conductor to another without losing any.

We now wish to construct a third streamline, and both the mathematical and visual interpretations we may make from the sketch will be greatly simplified if we draw this line starting from some point C chosen so that the same amount of flux is carried in the tube BC as is contained in AB. How do we choose the position of C?

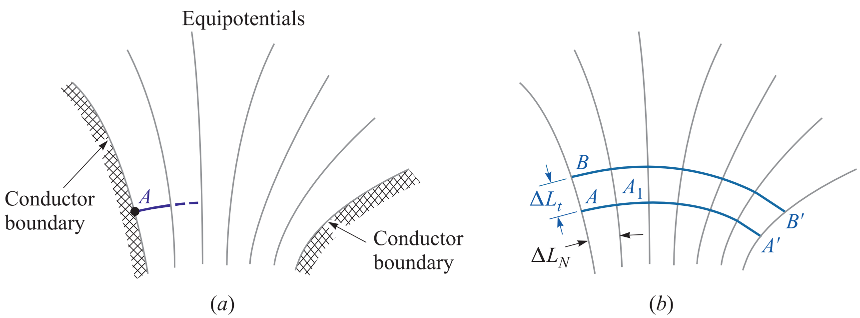

FIGURE 6.1
(*a*) Sketch of the equipotential surfaces between two conductors. The increment of potential between each of the two adjacent equipotentials is the same. (*b*) One flux line has been drawn from A to A', and a second from B to B'.

The electric field intensity at the midpoint of the line joining A to B may be found approximately by assuming a value for the flux in the tube AB, say $\Delta\Psi$, which allows us to express the electric flux density by $\Delta\Psi/\Delta L_t$, where the depth of the tube into the paper is 1 m and ΔL_t is the length of the line joining A to B. The magnitude of E is then

$$E = \frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta L_t}$$

However, we may also find the magnitude of the electric field intensity by dividing the potential difference between points A and A_1, lying on two adjacent equipotential surfaces, by the distance from A to A_1. If this distance is designated ΔL_N and an increment of potential between equipotentials of ΔV is assumed, then

$$E = \frac{\Delta V}{\Delta L_N}$$

This value applies most accurately to the point at the middle of the line segment from A to A_1, while the previous value was most accurate at the midpoint of the line segment from A to B. If, however, the equipotentials are close together (ΔV small) and the two streamlines are close together ($\Delta\Psi$ small), the two values found for the electric field intensity must be approximately equal,

$$\frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta L_t} = \frac{\Delta V}{\Delta L_N} \tag{1}$$

Throughout our sketch we have assumed a homogeneous medium (ϵ constant), a constant increment of potential between equipotentials (ΔV constant), and a constant amount of flux per tube ($\Delta\Psi$ constant). To satisfy all these conditions, (1) shows that

$$\boxed{\frac{\Delta L_t}{\Delta L_N} = \text{constant} = \frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta V}} \tag{2}$$

A similar argument might be made at any point in our sketch, and we are therefore led to the conclusion that a constant ratio must be maintained between the distance between streamlines as measured along an equipotential, and the distance between equipotentials as measured along a streamline. It is this *ratio* which must have the same value at every point, not the individual lengths. Each length must decrease in regions of greater field strength, because ΔV is constant.

The simplest ratio we can use is unity, and the streamline from B to B' shown in Fig. 6.1*b* was started at a point for which $\Delta L_t = \Delta L_N$. Since the ratio of these distances is kept at unity, the streamlines and equipotentials divide the field-containing region into curvilinear squares, a term implying a planar geometric figure which differs from a true square in having slightly curved and slightly unequal sides but which approaches a square as its dimensions decrease.

Those incremental surface elements in our three coordinate systems which are planar may also be drawn as curvilinear squares.

We may now rapidly sketch in the remainder of the streamlines by keeping each small box as square as possible. The complete sketch is shown in Fig. 6.2.

The only difference between this example and the production of a field map using the method of curvilinear squares is that the intermediate potential surfaces are not given. The streamlines and equipotentials must both be drawn on an original sketch which shows only the conductor boundaries. Only one solution is possible, as we shall prove later by the uniqueness theorem for Laplace's equation, and the rules we have outlined above are sufficient. One streamline is begun, an equipotential line is roughed in, another streamline is added, forming a curvilinear square, and the map is gradually extended throughout the desired region. Since none of us can ever expect to be perfect at this, we shall soon find that we can no longer make squares and also maintain right-angle corners. An error is accumulating in the drawing, and our present troubles should indicate the nature of the correction to make on some of the earlier work. It is usually best to start again on a fresh drawing, with the old one available as a guide.

The construction of a useful field map is an art; the science merely furnishes the rules. Proficiency in any art requires practice. A good problem for beginners is the coaxial cable or coaxial capacitor, since all the equipotentials are circles, while the flux lines are straight lines. The next sketch attempted should be two parallel circular conductors, where the equipotentials are again circles, but with different centers. Each of these is given as a problem at the end of the chapter, and the accuracy of the sketch may be checked by a capacitance calculation as outlined below.

Fig. 6.3 shows a completed map for a cable containing a square inner conductor surrounded by a circular conductor. The capacitance is found from $C = Q/V_0$ by replacing Q by $N_Q \Delta Q = N_Q \Delta \Psi$, where N_Q is the number of flux tubes joining the two conductors, and letting $V_0 = N_V \Delta V$, where N_V is the number of potential increments between conductors,

$$C = \frac{N_Q \Delta Q}{N_V \Delta V}$$

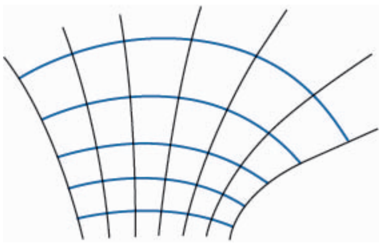

FIGURE 6.2
The remainder of the streamlines have been added to Fig. 6.1*b* by beginning each new line normally to the conductor and maintaining curvilinear squares throughout the sketch.

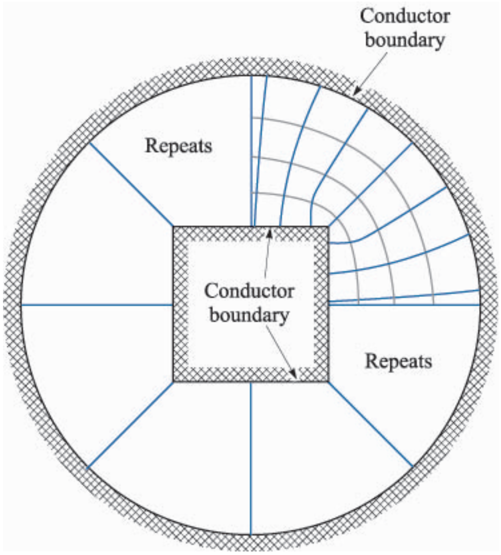

FIGURE 6.3
An example of a curvilinear-square field map. The side of the square is two thirds the radius of the circle. $N_V = 4$ and $N_Q = 8 \times 3.25 \times 26$, and therefore $C = \epsilon_0 N_Q / N_V = 57.6\,\text{pF/m}$.

and then using (2),

$$\boxed{C = \frac{N_Q}{N_V} \epsilon \frac{\Delta L_t}{\Delta L_N} = \epsilon \frac{N_Q}{N_V}} \qquad (3)$$

since $\Delta L_t / \Delta L_N = 1$. The determination of the capacitance from a flux plot merely consists of counting squares in two directions, between conductors and around either conductor. From Fig. 6.3 we obtain

$$C = \epsilon_0 \frac{8 \times 3.25}{4} = 57.6 \quad \text{pF/m}$$

Ramo, Whinnery, and Van Duzer have an excellent discussion with examples of the construction of field maps by curvilinear squares. They offer the following suggestions:[1]

1. Plan on making a number of rough sketches, taking only a minute or so apiece, before starting any plot to be made with care. The use of transparent paper over the basic boundary will speed up this preliminary sketching.
2. Divide the known potential difference between electrodes into an equal number of divisions, say four or eight to begin with.

[1] By permission from S. Ramo, J. R. Whinnery, and T. Van Duzer, pp. 51–52. See Suggested References at the end of Chap. 5. Curvilinear maps are discussed on pp. 50–52.

3. Begin the sketch of equipotentials in the region where the field is known best, as for example in some region where it approaches a uniform field. Extend the equipotentials according to your best guess throughout the plot. Note that they should tend to hug acute angles of the conducting boundary and be spread out in the vicinity of obtuse angles of the boundary.
4. Draw in the orthogonal set of field lines. As these are started, they should form curvilinear squares, but, as they are extended, the condition of orthogonality should be kept paramount, even though this will result in some rectangles with ratios other than unity.
5. Look at the regions with poor side ratios and try to see what was wrong with the first guess of equipotentials. Correct them and repeat the procedure until reasonable curvilinear squares exist throughout the plot.
6. In regions of low field intensity, there will be large figures, often of five or six sides. To judge the correctness of the plot in this region, these large units should be subdivided. The subdivisions should be started back away from the region needing subdivision, and each time a flux tube is divided in half, the potential divisions in this region must be divided by the same factor.

✔ **D6.1.** Figure 6.4 shows the cross section of two circular cylinders at potentials of 0 and 60 V. The axes are parallel and the region between the cylinders is air-filled.

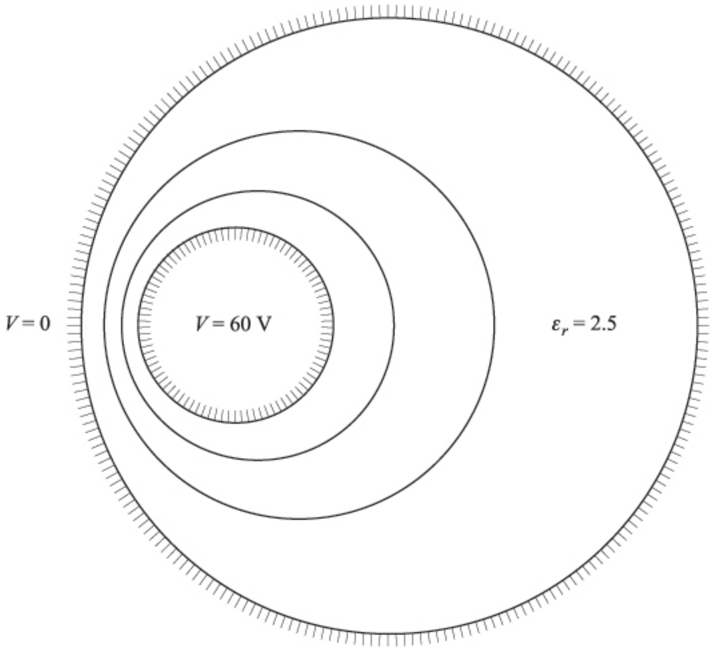

FIGURE 6.4
See Prob. D6.1.

Equipotentials at 20 V and 40 V are also shown. Prepare a curvilinear-square map on the figure and use it to establish suitable values for: (*a*) the capacitance per meter length; (*b*) E at the left side of the 60-V conductor if its true radius is 2 mm; (*c*) ρ_S at that point.

Ans. 69 pF/m; 60 kV/m; 550 nC/m^2

6.2 THE ITERATION METHOD

In potential problems where the potential is completely specified on the boundaries of a given region, particularly problems in which the potential does not vary in one direction (i.e., two-dimensional potential distributions) there exists a pencil-and-paper repetitive method which is capable of yielding any desired accuracy. Digital computers should be used when the value of the potential is required with high accuracy; otherwise, the time required is prohibitive except in the simplest problems. The iterative method, to be described below, is well suited for calculation by any digital computer.

Let us assume a two-dimensional problem in which the potential does not vary with the z coordinate and divide the interior of a cross section of the region where the potential is desired into squares of length h on a side. A portion of this region is shown in Fig. 6.5. The unknown values of the potential at five adjacent points are indicated as V_0, V_1, V_2, V_3, and V_4. If the region is charge-free and contains a homogeneous dielectric, then $\nabla \cdot \mathbf{D} = 0$ and $\nabla \cdot \mathbf{E} = 0$, from which we have, in two dimensions,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

But the gradient operation gives $E_x = -\partial V/\partial x$ and $E_y = -\partial V/\partial y$, from which we obtain[2]

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Approximate values for these partial derivatives may be obtained in terms of the assumed potentials, or

$$\left.\frac{\partial V}{\partial x}\right|_a \doteq \frac{V_1 - V_0}{h}$$

and

$$\left.\frac{\partial V}{\partial x}\right|_c \doteq \frac{V_0 - V_3}{h}$$

[2] This is Laplace's equation in two dimensions. The three-dimensional form will be derived in the following chapter.

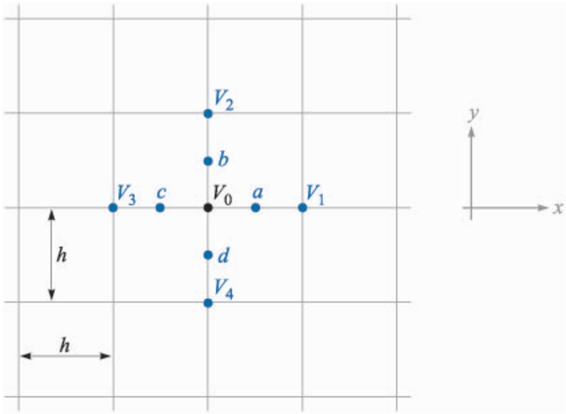

FIGURE 6.5
A portion of a region containing a two-dimensional potential field, divided into squares of side h. The potential V_0 is approximately equal to the average of the potentials at the four neighboring points.

from which

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_0 \doteq \frac{\left.\dfrac{\partial V}{\partial x}\right|_a - \left.\dfrac{\partial V}{\partial x}\right|_c}{h} \doteq \frac{V_1 - V_0 - V_0 + V_3}{h^2}$$

and similarly,

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_0 \doteq \frac{V_2 - V_0 - V_0 + V_4}{h^2}$$

Combining, we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \doteq \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{h^2} = 0$$

or

$$\boxed{V_0 \doteq \tfrac{1}{4}(V_1 + V_2 + V_3 + V_4)} \tag{4}$$

The expression becomes exact as h approaches zero, and we shall write it without the approximation sign. It is intuitively correct, telling us that the potential is the average of the potential at the four neighboring points. The iterative method merely uses (4) to determine the potential at the corner of every square subdivision in turn, and then the process is repeated over the entire region as many times as is necessary until the values no longer change. The method is best shown in detail by an example.

For simplicity, consider a square region with conducting boundaries (Fig. 6.6). The potential of the top is 100 V and that of the sides and bottom is zero. The problem is two-dimensional, and the sketch is a cross section of the physical configuration. The region is divided first into 16 squares, and some estimate of

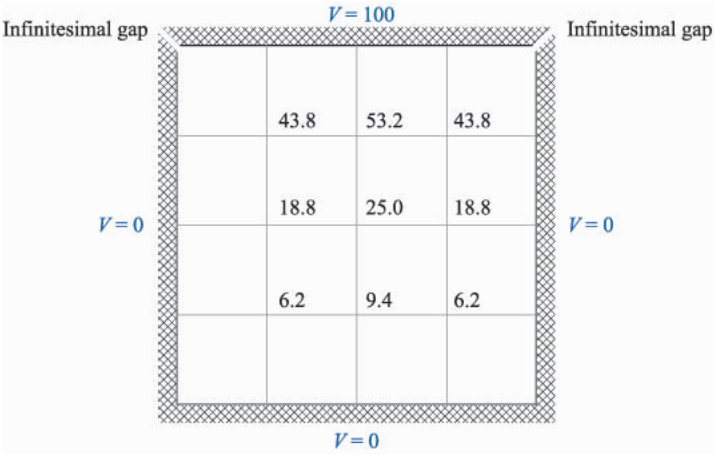

FIGURE 6.6
Cross section of a square trough with sides and bottom at zero potential and top at 100 V. The cross section has been divided into 16 squares, with the potential estimated at every corner. More accurate values may be determined by using the iteration method.

the potential must now be made at every corner before applying the iterative method. The better the estimate, the shorter the solution, although the final result is independent of these initial estimates. When the computer is used for iteration, the initial potentials are usually set equal to zero to simplify the program. Reasonably accurate values could be obtained from a rough curvilinear-square map, or we could apply (4) to the large squares. At the center of the figure the potential estimate is then $\frac{1}{4}(100 + 0 + 0 + 0) = 25.0$

The potential may now be estimated at the centers of the four double-sized squares by taking the average of the potentials at the four corners or applying (4) along a diagonal set of axes. Use of this "diagonal average" is made only in preparing initial estimates. For the two upper double squares, we select a potential of 50 V for the gap (the average of 0 and 100), and then $V = \frac{1}{4}(50 + 100 + 25 + 0) = 43.8$ (to the nearest tenth of a volt[3]), and for the lower ones,

$$V = \frac{1}{4}(0 + 25 + 0 + 0) = 6.2$$

The potential at the remaining four points may now be obtained by applying (4) directly. The complete set of estimated values is shown in Fig. 6.6.

The initial traverse is now made to obtain a corrected set of potentials, beginning in the upper left corner (with the 43.8 value, not with the boundary

[3] When rounding off a decimal ending exactly with a five, the preceding digit should be made *even* (e.g., 42.75 becomes 42.8 and 6.25 becomes 6.2). This generally ensures a random process leading to better accuracy than would be obtained by always increasing the previous digit by 1.

where the potentials are known and fixed), working across the row to the right, and then dropping down to the second row and proceeding from left to right again. Thus the 43.8 value changes to $\frac{1}{4}(100 + 53.2 + 18.8 + 0) = 43.0$. The best or newest potentials are always used when applying (4), so both points marked 43.8 are changed to 43.0, because of the evident symmetry, and the 53.2 value becomes $\frac{1}{4}(100 + 43.0 + 25.0 + 43.0) = 52.8$.

Because of the symmetry, little would be gained by continuing across the top line. Each point of this line has now been improved once. Dropping down to the next line, the 18.8 value becomes

$$V = \tfrac{1}{4}(43.0 + 25.0 + 6.2 + 0) = 18.6$$

and the traverse continues in this manner. The values at the end of this traverse are shown as the top numbers in each column of Fig. 6.7. Additional traverses must now be made until the value at each corner shows no change. The values for the successive traverses are usually entered below each other in column form, as shown in Fig. 6.7, and the final value is shown at the bottom of each column. Only four traverses are required in this example.

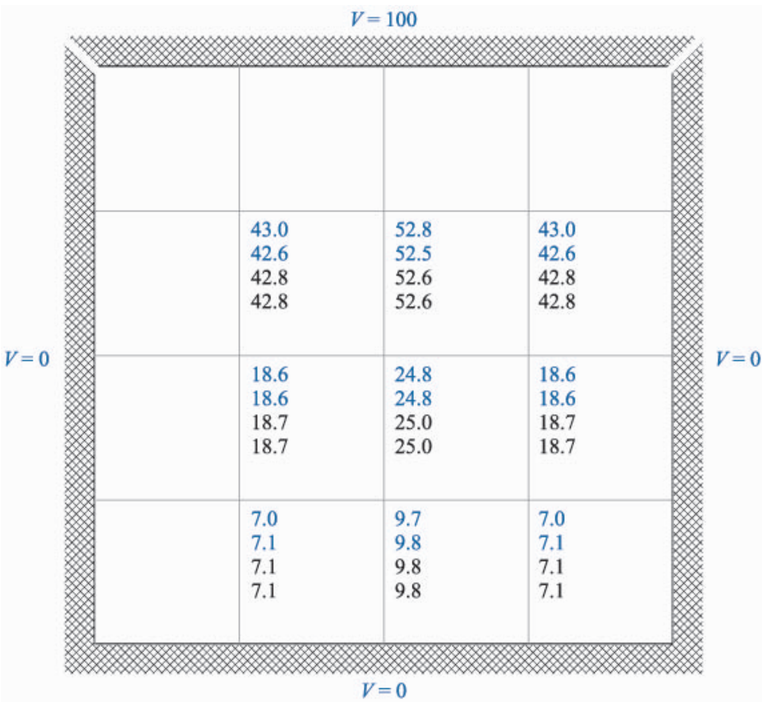

FIGURE 6.7
The results of each of the four necessary traverses of the problem of Fig. 6.5 are shown in order in the columns. The final values, unchanged in the last traverse, are at the bottom of each column.

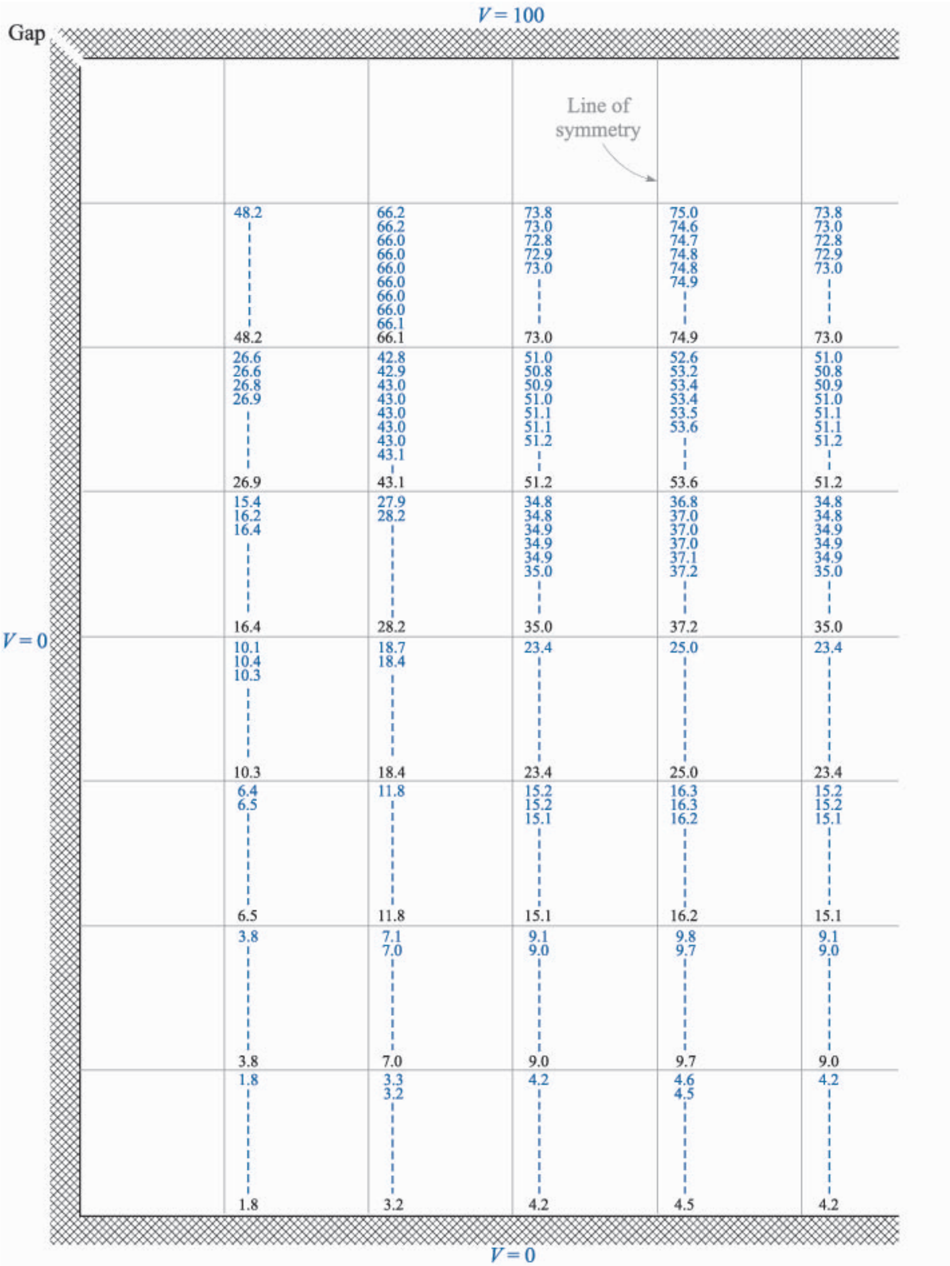

FIGURE 6.8
The problem of Figs. 6.6 and 6.7 is divided into smaller squares. Values obtained on the nine successive traverses are listed in order in the columns.

If each of the nine initial values were set equal to zero, it is interesting to note that ten traverses would be required. The cost of having a computer do these additional traverses is probably much less than the cost of the programming necessary to make decent initial estimates.

Since there is a large difference in potential from square to square, we should not expect our answers to be accurate to the tenth of a volt shown (and perhaps not to the nearest volt). Increased accuracy comes from dividing each square into four smaller squares, not from finding the potential to a larger number of significant figures at each corner.

In Fig. 6.8, which shows only one of the symmetrical halves plus an additional column, this subdivision is accomplished, and the potential at the newly created corners is estimated by applying (4) directly where possible and diagonally when necessary. The set of estimated values appears at the top of each column, and the values produced by the successive traverses appear in order below. Here nine sets of values are required, and it might be noted that no values change on the last traverse (a necessary condition for the *last* traverse), and only one value changes on each of the preceding three traverses. No value in the bottom four rows changes after the second traverse; this results in a great saving in time, for if none of the four potentials in (4) changes, the answer is of course unchanged.

For this problem, it is possible to compare our final values with the exact potentials, obtained by evaluating some infinite series, as discussed at the end of the following chapter. At the point for which the original estimate was 53.2, the final value for the coarse grid was 52.6, the final value for the finer grid was 53.6, and the final value for a 16×16 grid is 53.93 V to two decimals, using data obtained with the following Fortran program:

```
DIMENSION A (17,17),B(17,17)
DO 6 I=2,17
DO 5 J=1,17
A(I,J)=0.
5 CONTINUE
6 CONTINUE
DO 9 J=2,16
A(I,J)=100.
9 CONTINUE
A(1,1)=50.
A(1,17)=50.
DO 16 I=2,16
DO 15 J=2,16
A(I,J)=(A(I,J-1)+A(I-1,J)+A(I,J+1)+A(I+1,J))/4.
15 CONTINUE
16 CONTINUE
DO 23 I=2,16
DO 22 J=2,16
```

```
19  C=(A(I,J-1)+A(I-1,J)+A(I,J+1)+A(I+1,J))/4.
20  B(I,J)=A(I,J)-C
21  IF((ABS(B(I,J))-.00001).GT.0.) GO TO 12
22  CONTINUE
23  CONTINUE
24  WRITE(6,25)((A(I,J),J=1,17),I=1,17)
25  FORMAT (1H0,17F7.2)
26  STOP
27  END
```

Line 21 shows that the iteration is continued until the difference between successive traverses is less than 10^{-5}.

The exact potential obtained by a Fourier expansion is 54.05 V to two decimals. Two other points are also compared in tabular form, as shown in Table 6.1.

Computer flowcharts and programs for iteration solutions are given in Chap. 24 of Boast[4] and in Chap. 2 and the appendix of Silvester.[5]

Very few electrode configurations have a square or rectangular cross section that can be neatly subdivided into a square grid. Curved boundaries, acute- or obtuse-angled corners, reentrant shapes, and other irregularities require slight modifications of the basic method. An important one of these is described in Prob. 10 at the end of this chapter, and other irregular examples appear as Probs. 7, 9, and 11.

A refinement of the iteration method is known as the *relaxation method*. In general it requires less work but more care in carrying out the arithmetical steps.[6]

✔ **D6.2.** In Fig. 6.9, a square grid is shown within an irregular potential trough. Using the iteration method to find the potential to the nearest volt, determine the final value at: (*a*) point *a*; (*b*) point *b*; (*c*) point *c*.

Ans. 18 V; 46 V; 91 V

TABLE 6.1

Original estimate	53.2	25.0	9.4
4 × 4 grid	52.6	25.0	9.8
8 × 8 grid	53.6	25.0	9.7
16 × 16 grid	53.93	25.00	9.56
Exact	54.05	25.00	9.54

[4] See Suggested References at the end of Chap. 2.

[5] See Suggested References at the end of this chapter.

[6] A detailed description appears in Scarborough, and the basic procedure and one example are in an earlier edition of Hayt. See Suggested References at the end of the chapter.

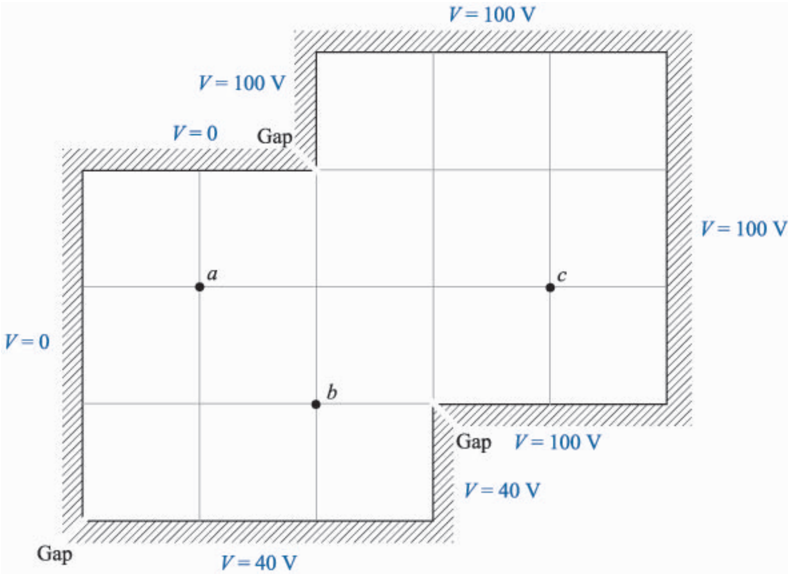

FIGURE 6.9
See Prob. D6.2.

6.3 CURRENT ANALOGIES

Several experimental methods depend upon an analogy between current density in conducting media and electric flux density in dielectric media. The analogy is easily demonstrated, for in a conducting medium Ohm's law and the gradient relationship are, for direct currents only,

$$\mathbf{J} = \sigma \mathbf{E}_\sigma$$

$$\mathbf{E}_\sigma = -\nabla V_\sigma$$

whereas in a homogeneous dielectric

$$\mathbf{D} = \epsilon \mathbf{E}_\epsilon$$

$$\mathbf{E}_\epsilon = -\nabla V_\epsilon$$

The subscripts serve to identify the analogous problems. It is evident that the potentials V_σ and V_ϵ, the electric field intensities $\mathbf{E}_\sigma$ and $\mathbf{E}_\epsilon$, the conductivity and permittivity σ and ϵ, and the current density and electric flux density $\mathbf{J}$ and $\mathbf{D}$ are analogous in pairs. Referring to a curvilinear-square map, we should interpret flux tubes as current tubes, each tube now carrying an incremental current which cannot leave the tube.

Finally, we must look at the boundaries. What is analogous to a conducting boundary which terminates electric flux normally and is an equipotential surface? The analogy furnishes the answer, and we see that the surface must termi-

nate current density normally and again be an equipotential surface. This is the surface of a *perfect* conductor, although in practice it is necessary only to use one whose conductivity is many times that of the conducting medium.

Therefore, if we wished to find the field within a coaxial capacitor, which, as we have seen several times before, is a portion of the field of an infinite line charge, we might take two copper cylinders and fill the region between them with, for convenience, an electrolytic solution. Applying a potential difference betwen the cylinders, we may use a probe to establish the potential at any intermediate point, or to find all those points having the same potential. This is the essence of the electrolytic trough or tank. The greatest advantage of this method lies in the fact that it is not limited to two-dimensional problems. Practical suggestions for the construction and use of the tank are given in many places.[7]

The determination of capacitance from electrolytic-trough measurements is particularly easy. The total current leaving the more positive conductor is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = \sigma \oint_S \mathbf{E}_\sigma \cdot d\mathbf{S}$$

where the closed surface integral is taken over the entire conductor surface. The potential difference is given by the negative line integral from the less to the more positive plate,

$$V_{\sigma 0} = -\int \mathbf{E}_\sigma \cdot d\mathbf{L}$$

and the total resistance is therefore

$$R = \frac{V_{\sigma 0}}{I} = \frac{-\int \mathbf{E}_\sigma \cdot d\mathbf{L}}{\sigma \oint_S \mathbf{E}_\sigma \cdot d\mathbf{S}}$$

The capacitance is given by the ratio of the total charge to the potential difference,

$$C = \frac{Q}{V_{\epsilon 0}} = \frac{\epsilon \oint_S \mathbf{E}_\epsilon \cdot d\mathbf{S}}{-\int \mathbf{E}_\epsilon \cdot d\mathbf{L}}$$

We now invoke the analogy by letting $V_{\epsilon 0} = V_{\sigma 0}$ and $\mathbf{E}_\epsilon = \mathbf{E}_\sigma$. The result is

$$\boxed{RC = \frac{\epsilon}{\sigma}} \tag{5}$$

Knowing the conductivity of the electrolyte and the permittivity of the dielectric, we may determine the capacitance by a simple resistance measurement.

A simpler technique is available for two-dimensional problems. Conducting paper is used as the base on which the conducting boundaries are drawn with

[7] Weber is good. See Suggested References at the end of the chapter.

silver paint. In the case of the coaxial capacitor, we should draw two circles of radii ρ_A and ρ_B, $\rho_B > \rho_A$, extending the paint a small distance outward from ρ_B and inward from ρ_A to provide sufficient area to make a good contact with wires to an external potential source. A probe is again used to establish potential values between the circles.

Conducting paper is described in terms of its *sheet resistance* R_S. The sheet resistance is the resistance between opposite edges of a square. Since the fields are uniform in such a square, we may apply Eq. (13) from Chap. 5,

$$R = \frac{L}{\sigma S}$$

to the case of a square of conducting paper having a width w and a thickness t, where $w = L$,

$$R_S = \frac{L}{\sigma t L} = \frac{l}{\sigma t} \quad \Omega \tag{6}$$

Thus, if the conductive coating has a thickness of 0.2 mm and a conductivity of 2 S/m, its surface resistance is $1/(2 \times 0.2 \times 10^{-3}) = 2500\,\Omega$. The units of R_S are often given as "ohms per square" (but never as ohms per square meter).

Fig. 6.10 shows the silver-paint boundaries that would be drawn on the conducting paper to determine the capacitance of a square-in-a-circle transmission line like that shown in Fig. 6.3. The generator and detector often operate at 1 kHz to permit use of a more sensitive tuned detector or bridge.

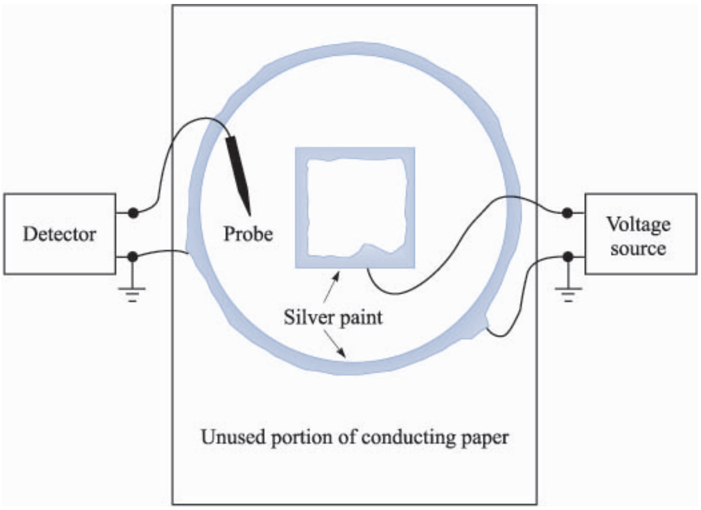

FIGURE 6.10
A two-dimensional, two-conductor problem similar to that of Fig. 6.3 is drawn on conducting paper. The probe may be used to trace out an equipotential surface.

✔ **D6.3.** If the conducting paper shown in Fig. 6.10 has a sheet resistance of 1800 Ω per square, find the resistance that would be measured between the opposite edges of: (*a*) a square 9 cm on a side; (*b*) a square 4.5 cm on a side; (*c*) a rectangle 3 cm by 9 cm, across the longer dimension; (*d*) a rectangle 3 cm by 9 cm, across the shorter dimension. (*e*) What resistance would be measured between an inner circle of 0.8-cm radius and an outer circle of 2-cm radius?

Ans. 1800 Ω; 1800 Ω; 5400 Ω; 600 Ω; 397 Ω

6.4 PHYSICAL MODELS

The analogy between the electric field and the gravitational field was mentioned several times previously and may be used to construct physical models which are capable of yielding solutions to electrostatic problems of complicated geometry. The basis of the analogy is simply this: in the electrostatic field the potential difference between two points is the difference in the potential energy of unit positive charges at these points, and in a uniform gravitational field the difference in the potential energy of point masses at two points is proportional to their difference in height. In other words,

$$\Delta W_E = Q\Delta V \quad \text{(electrostatic)}$$

$$\Delta W_G = Mg\Delta h \quad \text{(gravitational)}$$

where M is the point mass and g is the acceleration due to gravity, essentially constant at the surface of the earth. For the same energy difference, then,

$$\Delta V = \frac{Mg}{Q}\Delta h = k\Delta h$$

where k is the constant of proportionality. This shows the direct analogy between difference in potential and difference in elevation.

This analogy allows us to construct a physical model of a known two-dimensional potential field by fabricating a surface, perhaps from wood, whose elevation h above any point (x, y) located in the zero-elevation zero-potential plane is proportional to the potential at that point. Note that three-dimensional fields cannot be handled.

The field of an infinite line charge,

$$V = \frac{\rho_L}{2\pi\epsilon}\ln\frac{\rho_B}{\rho}$$

is shown on such a model in Fig. 6.11, which provides an accurate picture of the variation of potential with radius between ρ_A and ρ_B. The potential and elevation at ρ_B are conveniently set equal to zero.

Such a model may be constructed for any two-dimensional potential field and enables us to visualize the field a little better. The construction of the models themselves is enormously simplified, both physically and theoretically, by the use of rubber sheets. The sheet is placed under moderate tension and approximates

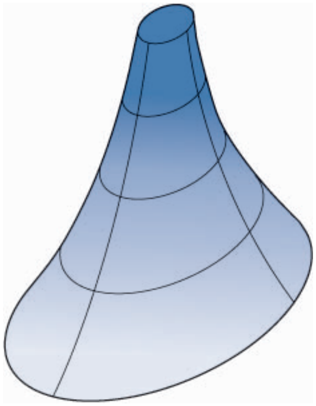

FIGURE 6.11
A model of the potential field of an infinite line charge. The difference in potential is proportional to the difference in elevation. Contour lines indicate equal potential increments.

closely the *elastic membrane* of applied mechanics. It can be shown[8] that the vertical displacement h of the membrane satisfies the second-order partial differential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

if the surface slope is small. We shall see in the next chapter that every potential field in a charge-free region also satisfies this equation, Laplace's equation in two dimensions,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We shall also prove a uniqueness theorem which assures us that if a potential solution in some specified region satisfies the above equation and also gives the correct potential on the boundaries of this region, then this solution is the only solution. Hence we need only force the elevation of the sheet to corresponding prescribed potential values on the boundaries, and the elevation at all other points is proportional to the potential.

For instance, the infinite-line-charge field may be displayed by recognizing the circular symmetry and fastening the rubber sheet at zero elevation around a circle by the use of a large clamping ring of radius ρ_B. Since the potential is constant at ρ_A, we raise that portion of the sheet to a greater elevation by pushing a cylinder of radius ρ_A up against the rubber sheet. The analogy breaks down for large surface slopes, and only a slight displacement at ρ_A is possible. The surface then represents the potential field, and marbles may be used to determine particle trajectories, in this case obviously radial lines as viewed from above.

[8] See, for instance, Spangenberg, pp. 75–76, in Suggested References at the end of the chapter.

There is also an analogy between electrostatics and hydraulics that is particularly useful in obtaining photographs of the streamlines or flux lines. This process is described completely by Moore in a number of publications[9] which include many excellent photographs.

✔ **D6.4.** A potential field, $V = 200(x^2 - 4y + 2)$ V, is illustrated by a plaster model with a scale of 1 vertical inch = 400 V; the horizontal dimensions are true. The region shown is $3 \leq x \leq 4$ m, $0 \leq y \leq 1$ m. In this region: (*a*) What is the maximum height of the model? (*b*) What is its minimum height? (*c*) What is the difference in height between points $A(x = 3.2, y = 0.5)$ and $B(3.8, 1)$? (*d*) What angle does the line connecting these two points make with the horizontal?

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8. Weber, E.: "Electromagnetic Fields," vol. I, John Wiley & Sons, Inc., New York, 1950. Experimental mapping methods are discussed in chap. 5.

[9] See Suggested References at the end of this chapter.

PROBLEMS

6.1 Construct a curvilinear-square map for a coaxial capacitor of 3-cm inner radius and 8-cm outer radius. These dimensions are suitable for the drawing. (*a*) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_R = 1$. (*b*) Calculate an exact value for the capacitance per unit length.

6.2 Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5-cm radius, separated a center-to-center distance of 13 cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_R = 1$.

6.3 Construct a curvilinear-square map of the potential field between two parallel circular cylinders, one of 4-cm radius inside one of 8-cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$\frac{2\pi\epsilon}{\cosh^{-1}\dfrac{a^2+b^2-D^2}{2ab}}$$

where a and b are the conductor radii and D is the axis separation.

6.4 A solid conducting cylinder of 4-cm radius is centered within a rectangular conducting cylinder with a 12-cm by 20-cm cross section. (*a*) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior. (*b*) Assume $\epsilon = \epsilon_0$ and estimate C per meter length.

6.5 The inner conductor of the transmission line shown in Fig. 6.12 has a square cross section $2a \times 2a$, while the outer square is $5a \times 5a$. The axes are displaced as shown. (*a*) Construct a good-sized drawing of this transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (*b*) Use your map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (*c*) How would the answer to part *b* change if $a = 0.6$ cm?

6.6 Let the inner conductor of the transmission line shown in Fig. 6.12 be at a potential of 100 V, while the outer is at zero potential. Construct a grid, $0.5a$ on a side, and use iteration to find V at a point that is a units above the upper right corner of the inner conductor. Work to the nearest volt.

6.7 Use the iteration method to estimate the potential at points x and y in the triangular trough of Fig. 6.13. Work only to the nearest volt.

6.8 Use iteration methods to estimate the potential at point x in the trough shown in Fig. 6.14. Working to the nearest volt is sufficient.

6.9 Using the grid indicated in Fig. 6.15, work to the nearest volt to estimate the potential at point A.

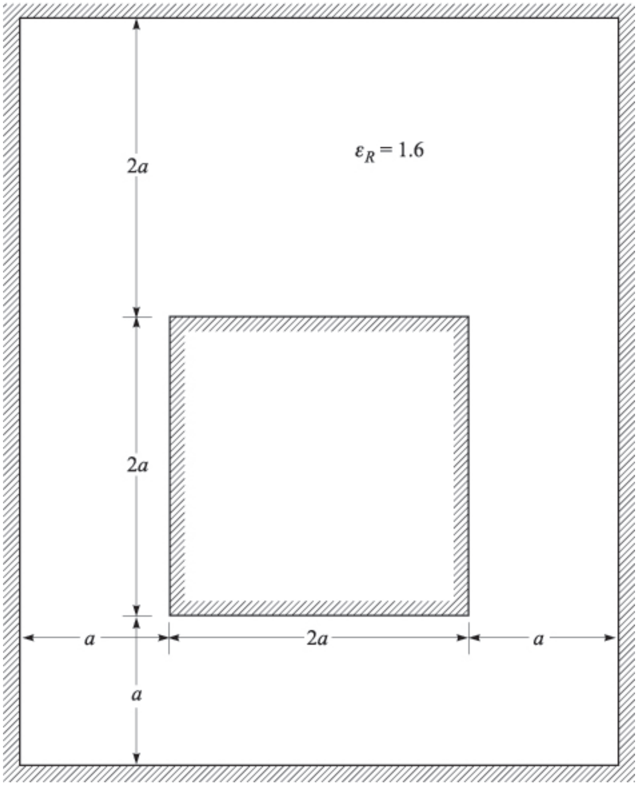

FIGURE 6.12
See Probs. 5, 6, and 14.

6.10 Conductors having boundaries that are curved or skewed usually do not permit every grid point to coincide with the actual boundary. Figure 6.16*a* illustrates the situation where the potential at V_0 is to be estimated in terms of V_1, V_2, V_3, V_4, and the unequal distances h_1, h_2, h_3, and h_4. (*a*) Show that

$$V_0 = \frac{V_1}{\left(1+\frac{h_1}{h_3}\right)\left(1+\frac{h_1 h_3}{h_4 h_2}\right)} + \frac{V_2}{\left(1+\frac{h_2}{h_4}\right)\left(1+\frac{h_2 h_4}{h_4 h_3}\right)} + \frac{V_3}{\left(1+\frac{h_3}{h_1}\right)\left(1+\frac{h_3 h_1}{h_2 h_4}\right)}$$

$$+ \frac{V_4}{\left(1+\frac{h_4}{h_2}\right)\left(1+\frac{h_4 h_2}{h_3 h_1}\right)};$$ (*b*) determine V_0 in Fig. 6.16*b*.

6.11 Consider the configuration of conductors and potentials shown in Fig. 6.17. Using the method described in Prob. 10, write an expression for V_0

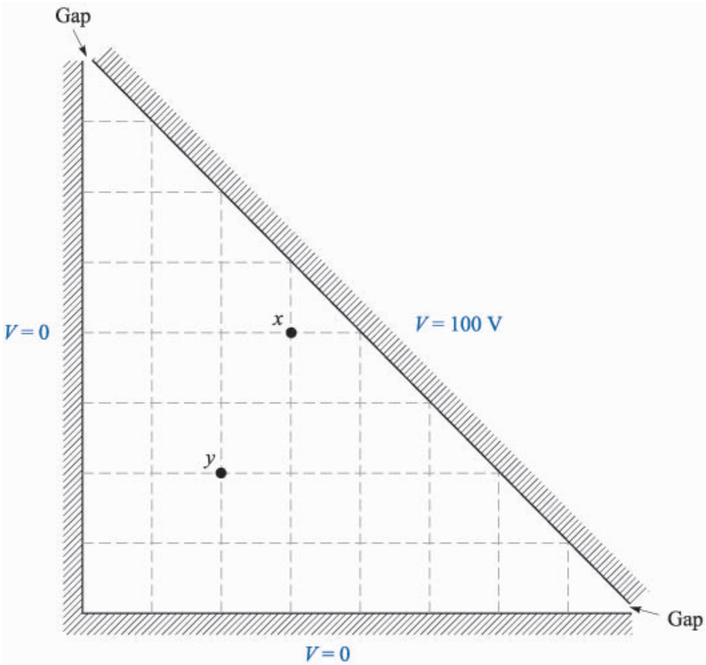

FIGURE 6.13
See Prob. 7.

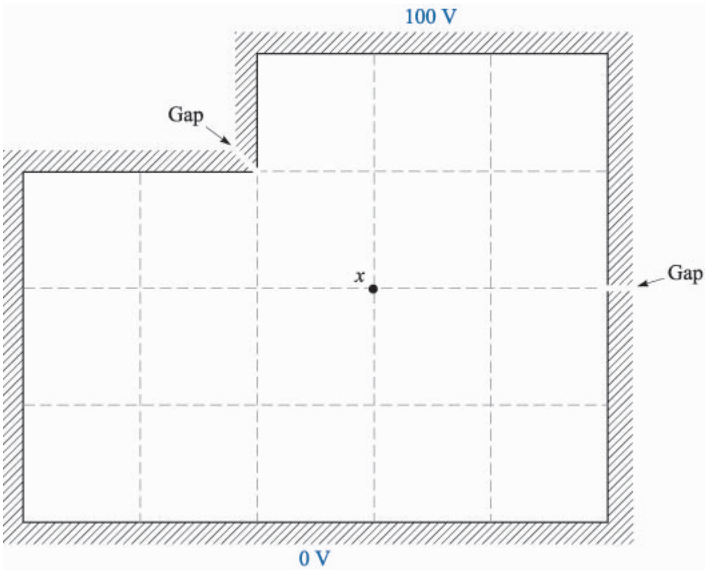

FIGURE 6.14
See Prob. 8.

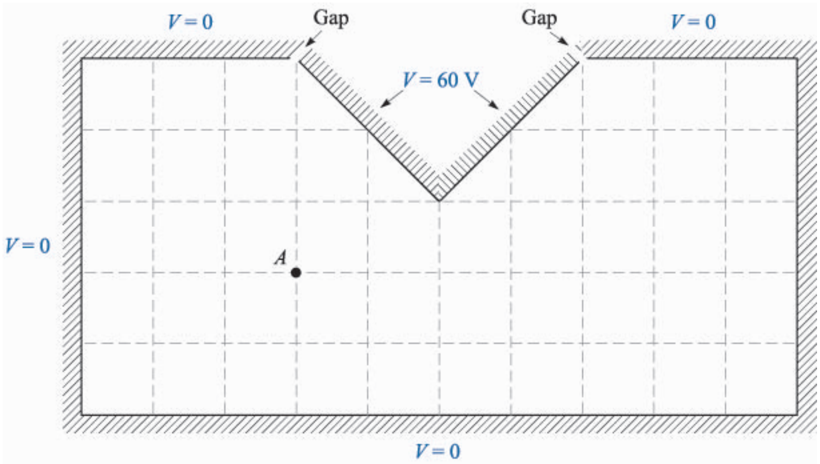

FIGURE 6.15
See Prob. 9.

in terms of V_1, V_2, V_3, and V_4 at point: (*a*) *x*; (*b*) *y*; (*c*) *z*. (*d*) Use the iteration method to estimate the potential at point *x*.

6.12 (*a*) After estimating potentials for the configuration of Fig. 6.18, use the iteration method with a square grid 1 cm on a side to find better estimates at the seven grid points. Work to the nearest volt. (*b*) Construct 0.5-cm grid, establish new rough estimates, and then use the iteration method on the 0.5-cm grid. Again work to the nearest volt. (*c*) Use the computer to obtain values for a 0.25-cm grid. Work to the nearest 0.1 V.

6.13 Perfectly conducting concentric spheres have radii of 2 and 6 cm. The region $2 < r < 3$ cm is filled with a solid conducting material for which

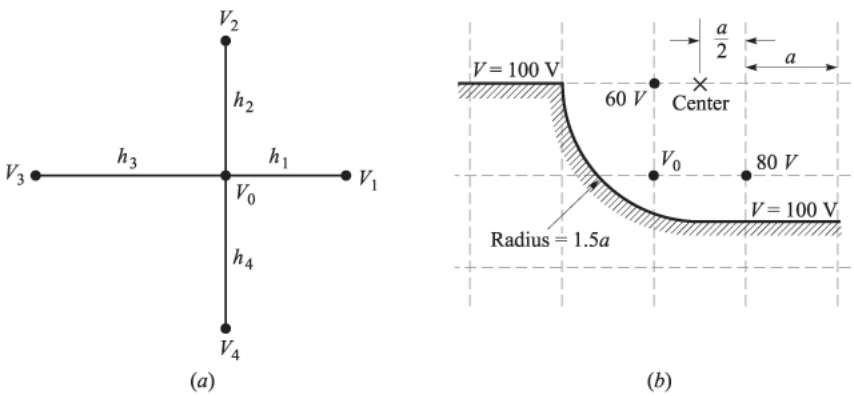

FIGURE 6.16
See Prob. 10.

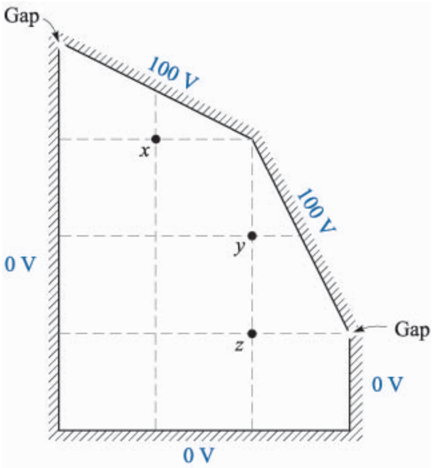

FIGURE 6.17
See Prob. 11.

$\sigma = 100$ S/m, while the portion for which $3 < r < 6$ cm has $\sigma = 25$ S/m. The inner sphere is held at 1 V while the outer is at $V = 0$. (*a*) Find E and J everywhere. (*b*) What resistance would be measured between the two spheres? (*c*) What is V at $r = 3$ cm?

6.14 The cross section of the transmission line shown in Fig. 6.12 is drawn on a sheet of conducting paper with metallic paint. The sheet resistance is 2000 Ω per square and the dimension a is 2 cm. (*a*) Assuming a result for Prob. 6*b* of 110 pF/m, what total resistance would be measured between the metallic conductors drawn on the conducting paper? (*b*) What would the total resistance be if $a = 2$ cm?

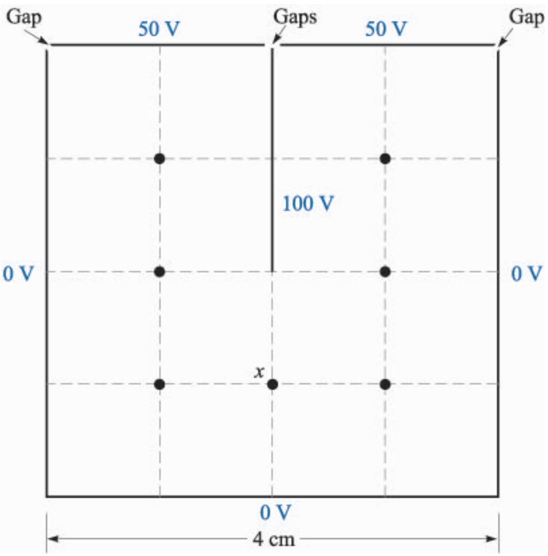

FIGURE 6.18
See Prob. 12.

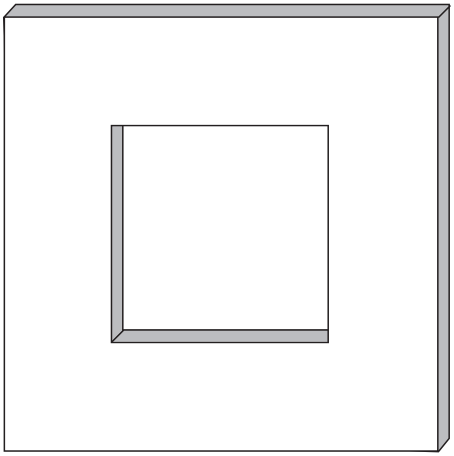

FIGURE 6.19
See Prob. 16.

6.15 Two concentric annular rings are painted on a sheet of conducting paper with a highly conducting metallic paint. The four radii are 1, 1.2, 3.5, and 3.7 cm. Connections made to the two rings show a resistance of 215 Ω between them. (*a*) What is R_S for the conducting paper? (*b*) If the conductivity of the material used as the surface of the paper is 2 S/m, what is the thickness of the coating?

6.16 The square washer shown in Fig. 6.19 is 2.4 mm thick and has outer dimensions of 2.5×2.5 cm and inner dimensions of 1.25×1.25 cm. The inside and outside surfaces are perfectly conducting. If the material has a conductivity of 6 S/m, estimate the resistance offered between the inner and outer surfaces (shown shaded in Fig. 6.19). A few curvilinear squares are suggested.

6.17 A two-wire transmission line consists of two parallel perfectly conducting cylinders, each having a radius of 0.2 mm, separated a center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_R = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. Calculate: (*a*) the magnitude of the charge per meter length on each wire; (*b*) the battery current.

6.18 A coaxial transmission line is modelled by the use of a rubber sheet having horizontal dimensions that are 100 times those of the actual line. Let the radial coordinate of the model be ρ_m. For the line itself, let the radial dimension be designated by ρ as usual; also, let $a = 0.6$ mm and $b = 4.8$ mm. The model is 8 cm in height at the inner conductor and zero at the outer. If the potential of the inner conductor is 100 V: (*a*) find the expression for $V(\rho)$. (*b*) Write the model height as a function of ρ.